Dynamic Programming Intro

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Dynamic Programming

- Outline:
    - General Principles
    - Easy Examples – Fibonacci, Licking Stamps
    - Meatier examples
        - RNA Structure prediction
        - Weighted interval scheduling
        - Maybe others

Some Algorithm Design Techniques, I

- General overall idea
  - Reduce solving a problem to a smaller problem or problems of the same type
- Greedy algorithms
  - Used when one needs to build something a piece at a time
  - Repeatedly make the greedy choice - the one that looks the best right away
    - e.g. closest pair in TSP search
  - Usually fast if they work (but often don't)

Some Algorithm Design Techniques, II

- Divide & Conquer
  - Reduce problem to one or more sub-problems of the same type
  - Typically, each sub-problem is at most a constant fraction of the size of the original problem
    - e.g. Mergesort, Binary Search, Strassen's Algorithm, Quicksort (kind of)

Some Algorithm Design Techniques, III

- Dynamic Programming
    - Give a solution of a problem using smaller sub-problems, e.g. a recursive solution
    - Useful when the same sub-problems show up again and again in the solution

Dynamic Programming History

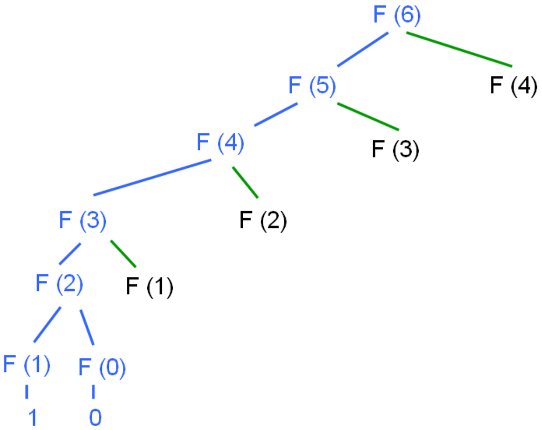
- Bellman. Pioneered the systematic study of dynamic programming in the 1950s.
- Etymology.
  - Dynamic programming = planning over time.
  - Secretary of Defense was hostile to mathematical research.
  - Bellman sought an impressive name to avoid confrontation.
    - "it's impossible to use dynamic in a pejorative sense"
    - "something not even a Congressman could object to"

Simple case:Computing Fibonacci Numbers

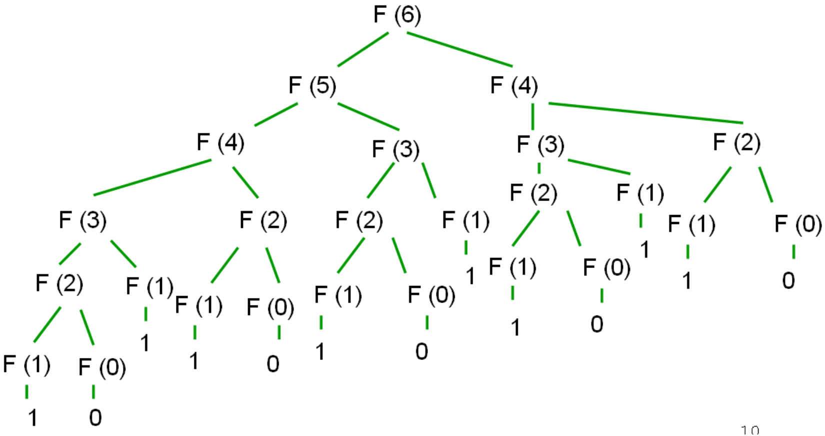
- Recall $F_n = F_{n-1} + F_{n-2}$ and $F_0 = 0, F_1 = 1$
- Recursive algorithm:

```
procedure FIBO($n$)
    if $n = 0$ then
        return 0
    else if $n = 1$ then
        return 1
    else
        return $Fibo(n-1) + Fibo(n-2)$
    end if
end procedure
```

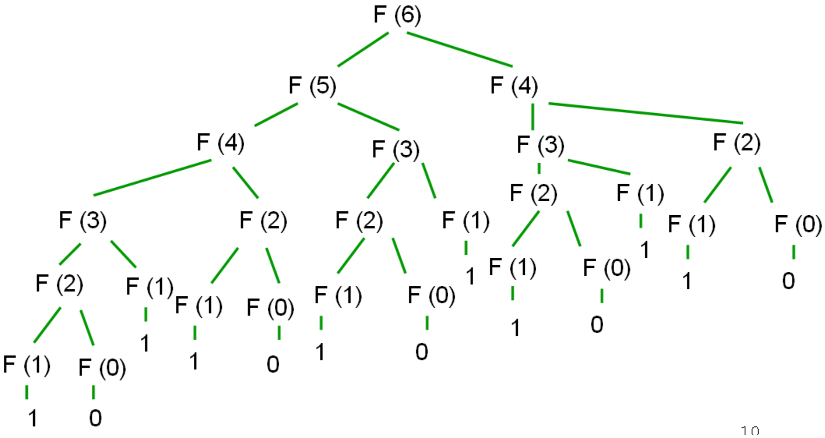
Recursive Call Tree



Recursive Call Tree



Recursive Call Tree



- Very slow, because of many repeated calculations!

Memo-ization (Caching)

- Remember all values from previous recursive calls
- Before recursive call, test to see if value has already been computed
- Dynamic Programming
  - could be memoized
  - or, convert recursion to iteration (top-down → bottom-up)

Fibonacci - Memoized Version

```
Initialize $F[i]$ undefined for all $i$
$F[0] \leftarrow 0$
$F[1] \leftarrow 1$
procedure FiboMem($n$)
    if $F[n]$ undefined then
        $F[n] \leftarrow FiboMem(n-1) + FiboMem(n-2)$
    end if
    return $F[n]$
end procedure
```

Fibonacci - Dynamic Programming Version

```
procedure FIBODP($n$)
    $F[0] \leftarrow 0$
    $F[1] \leftarrow 1$
    for $i = 2$ to $n$ do
        $F[i] \leftarrow F[i-1] + F[i-2]$
    end for
    return $F[n]$
end procedure
```

- for this problem, actually only need to keep last two entries, not full array ... but not a big difference

Dynamic Programming

- Useful when
    - Same recursive sub-problems occur repeatedly
    - Parameters of these recursive calls anticipated
    - The solution to whole problem can be solved without knowing the internal details of how the sub-problems are solved
        - “principle of optimality”

Making change

- Given:
    - Large supply of $1¢, 5¢, 10¢, 25¢, 50¢$ coins
    - An amount N
- Problem: choose fewest coins totaling N
- Cashier's (greedy) algorithm works:
    - Give as many as possible of the next biggest denomination

Licking Stamps

- Given:
  - Large supply of 5¢, 4¢, and 1¢ stamps
  - An amount N
- Problem: choose fewest stamps totaling N

How to Lick 27¢

# of 5 ¢ stamps	# 4 ¢ stamps	# 1 ¢ stamps	total # stamps
5	0	2	7
4	1	3	8
3	3	0	6

- Greed doesn't pay this time ...

A Simple Algorithm

- At most N stamps needed, etc.

```
for $a = 0, \dots, N$ do
    for $b = 0, \dots, N$ do
        for $c = 0, \dots, N$ do
            if $5a + 4b + c == N \&\& a + b + c$ is new min then
                retain (a,b,c)
            end if
        end for
    end for
end for
```

- Time: $O(N^3)$ (Not too hard to see some optimizations, but we're after bigger fish. . .)

The Magic Genie

- a useful way to think about dynamic programming (for me ...)
    - You can ask a magic genie as many thing as you want ...
    - ... but its power falls just short of your question. (Eg., it can onlyfigure how many stamps to use for up to 26¢)
    - Can you still use the genie to get a solution?

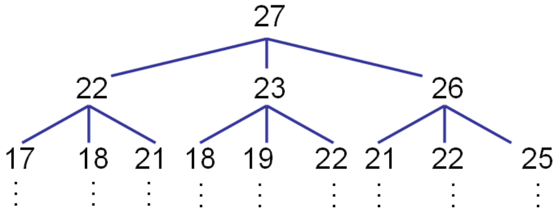
Better Idea

- Theorem: If last stamp in an opt sol has value v, then previous stamps are opt sol for $N - v$.
- Proof: if not, we could improve the solution for N by using opt for $N - v$.

```
for $i = 1$ to $n$ do
    $M(i) = min\left(\begin{cases} 0 & i = 0 \\ 1 + M(i-5) & i \geq 5 \\ 1 + M(i-4) & i \geq 4 \\ 1 + M(i-1) & i \geq 1 \end{cases}\right)$
end for
```

New Idea: Recursion

$$M(i) = min \left(\begin{cases} 0 & i = 0 \\ 1 + M(i-5) & i \geq 5 \\ 1 + M(i-4) & i \geq 4 \\ 1 + M(i-1) & i \geq 1 \end{cases} \right)$$

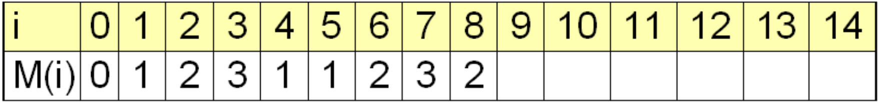


Time: $O(3^n)$

Another New Idea: Avoid Recomputation

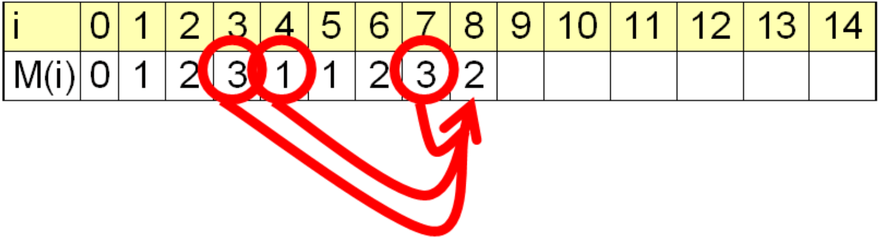
- Tabulate values of solved subproblems
    - Top-down: “memoization”
    - Bottom up:
      for i = 0, . . . , N do ;
- Time: O(N)

Finding How Many Stamps



i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
M(i)	0	1	2	3	1	1	2	3	2						

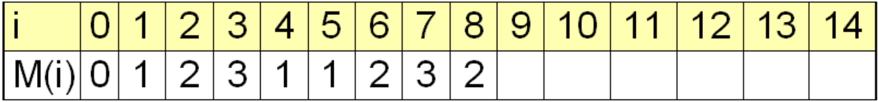
Finding How Many Stamps



i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
M(i)	0	1	2	3	1	1	2	3	2						

$$\begin{aligned} M[8] &= 1 + min(M[7], M[4], M[3]) \\ &= 1 + min(3, 1, 3) \\ M[8] &= 2 \end{aligned}$$

Finding Which Stamps: Trace-Back



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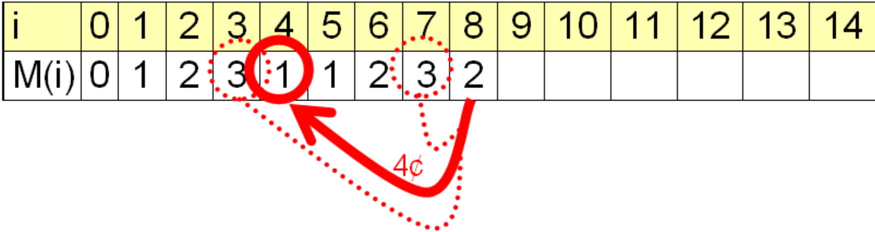

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M(i)	0	1	2	3	1	1	2	3	2						

Trace-Back

- Way 1: tabulate all
  - add data structure storing back-pointers
- Way 2: re-compute just what's needed

```
procedure TRACEBACK($i$)
    if $i = 0$ then return
    end if
    for $d \in \{1, 4, 5\}$ do
        if $M[i] == 1 + M[i - d]$ then break
        end if
    end for
    print $d$
    Traceback($i - d$)
end procedure
```

Complexity Note

- $O(N)$ is better than $O(N^3)$ or $O(3^{N/5})$
- But still <u>exponential</u> in input size ($\log N$ bits) (E.g., miserable if N is 64 bits – $c2^{64}$ steps & 2^{64} memory.)

Elements of Dynamic Programming

- What feature did we use?
- What should we look for to use again?
- Optimal Substructure
  Optimal solution contains optimal subproblems
    - A non-example: min (number of stamps mod 2)
- Repeated Subproblems The same subproblems arise in various ways