Huffman Codes

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Lecture Outline

Compression Example

- 100k file, 6 letter alphabet:
- File Size:
  - ASCII, 8 bits/char: 800kbits
  - $2^3 > 6$; 3 bits/char: 300kbits

letter	freq
a	.45
b	.13
c	.12
d	.16
e	.09
f	.05

Compression Example

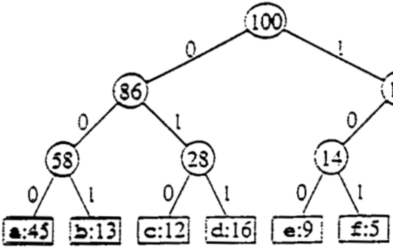
- 100k file, 6 letter alphabet:
- File Size:
  - ASCII, 8 bits/char: 800kbits
  - $2^3 > 6$; 3 bits/char: 300kbits
  - better: $(.45 + .13 + .12) * 2 + (.16 + .09 + .05) * 4 =$ 2.6bits / char
  - Optimal?

letter	freq	code
a	.45	00
b	.13	01
c	.12	10
d	.16	1100
e	.09	1101
f	.05	1110

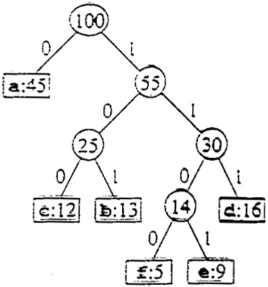
Data Compression

- Binary character code ("code")
  - each k-bit source string maps to unique code word (e.g. k=8)
  - "compression" alg: concatenate code words for successive k-bit "characters" of source
- Fixed/variable length codes
  - all code words equal length?
- Prefix codes
  - no code word is prefix of another (unique decoding)

Prefix Codes = Trees



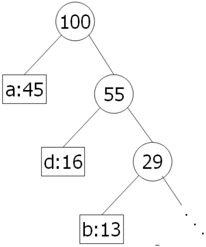
$$\underbrace{1\ 0\ 1}_{f}\ \underbrace{0\ 0\ 0}_{a}\ \underbrace{0\ 0\ 1}_{b}$$



$$\underbrace{1\ 1\ 0\ 0}_{f}\ \underbrace{0}_{a}\ \underbrace{1\ 0\ 1}_{b}$$

Greedy Idea #1

letter	freq
a	.45
b	.13
c	.12
d	.16
e	.09
f	.05



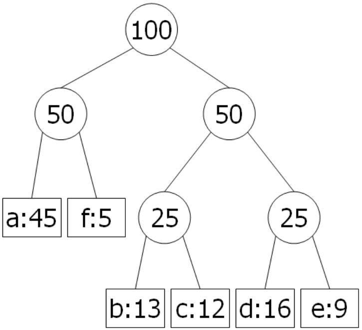
- Put most frequent under root, then recurse
- Too greedy: unbalanced tree
- $.45 * 1 + .16 * 2 + \ldots = 2.34$
- not too bad, but imagine if all freqs were $1/6$
  $(1 + 2 + 3 + 4 + 5 + 5)/6 = 3.33$

Greedy Idea #2

- Divide letters into 2 groups, with $\approx 50\%$ weight in each;recurse(Shannon-Fano code)
- Again, not terrible:

$$2 * .5 + 3 * .5 = 2.5$$

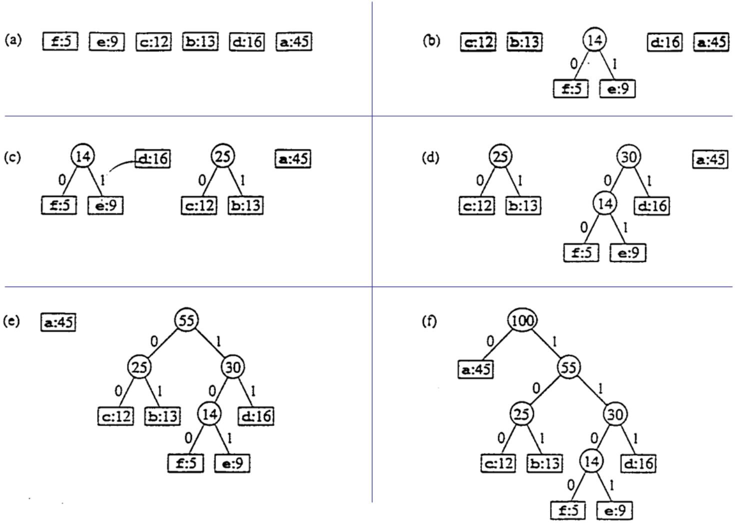
- But this tree can easily be improved! (How?)



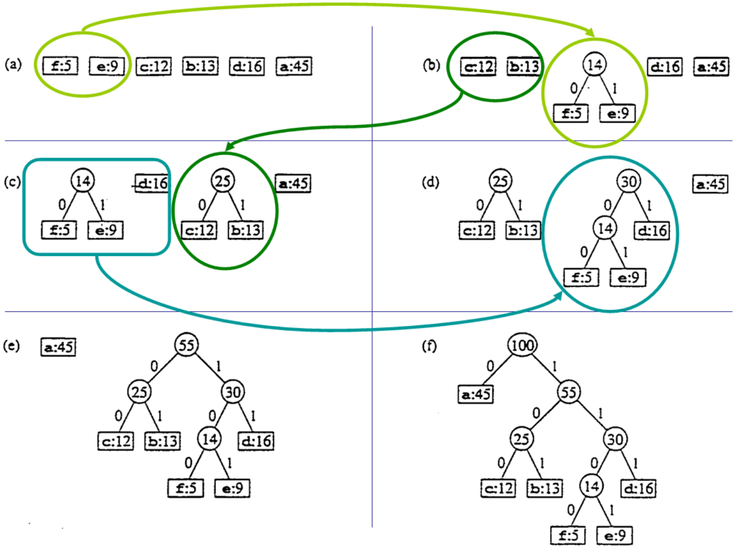
Greedy idea #3

- Group least frequent letters near bottom

Huffman example



Huffman example



Huffman's Algorithm (1952)

```
Insert node for each letter into priority queue by freq
while queue length > 1 do
    Remove smallest 2 nodes, call them x, y
    Make new node z with children x, y.
    f(z) = f(x) + f(y)
    Insert z into queue
end while
```

- Analysis: O(n) heap ops: O(n log n)
- Goal: Minimize $B(T) = \sum freq(c) * depth(c)$
- Correctness: ???

Correctness Strategy

- Optimal solution may not be unique, so cannot prove that greedy gives the only possible answer.
- Instead, show that greedy's solution is as good as any.

Inversions

- A pair of leaves x, y is in an inversion if
  $depth(x) \geq depth(y)$
  and
  $freq(x) \geq freq(y)$
- Claim: if we flip an inversion, cost never increases.

$$\underbrace{(d(x) * f(x) + d(y) * f(y))}_{\text{before}} - \underbrace{(d(x) * f(y) + d(y) * f(x))}_{\text{after}} =$$

$$(d(x) - d(y)) * (f(x) - f(y)) \geq 0$$

Lemma 1: "Greedy Choice Property"

- The 2 least frequent letters might as well be siblings at deepest level
  - Let a be least freq, b 2nd
  - Let u, v be siblings at max depth, $f(u) \leq f(v)$ (why must they exist?)
  - Then (a, u) and (b, v) are inversions. Swap them.

Lemma 2

- Let (C, f) be a problem instance: C an n-letter alphabet with letter frequencies $f(c)$ for $c \in C$.
- For any $x, y \in C$, let C' be the $(n-1)$ letter alphabet $C - \{x, y\} \cup \{z\}$.
- For all $c \in C', c \neq z$, let $f'(c) = f(c)$. And let $f'(z) = f(x) + f(y)$.
- Let T' be an optimal tree for (C', f').
- Then create tree T by adding x, y as children of z in T'.
- T is optimal for (C, f) among all trees having x, y as siblings

Proof of Lemma 2

Proof.

$$
\begin{aligned}
B(T) &= \sum_{c \in C} d_T(c) * f(c) \\
B(T) - B(T') &= d_T(x) * (f(x) + f(y)) - d_{T'}(z) * f'(z) \\
&= (d_{T'}(z) + 1) * f'(z) - d_{T'}(z) * f'(z) \\
&= f'(z)
\end{aligned}
$$

Suppose $\widehat{T}$ (having x, y siblings) is better than T (eg., $B(\widehat{T}) < B(T)$). Collapse x, y to form $\widehat{T'}$. As above, $B(\widehat{T}) - B(\widehat{T'}) = f'(z)$. Then $B(\widehat{T'}) = B(\widehat{T}) - f'(z) < B(T) - f'(z) = B(T')$. This contradicts the optimality of T'. $\square$

Theorem: Huffman gives optimal codes

Proof.

By Induction on $|C|$.

- Basis: $n = 1, 2$ – immediate
- Induction: $n > 2$
  - Let x, y be least frequent
  - Form $C', f', \&z$, as above
  - By induction, T' is opt for (C', f')
  - By lemma 2, T created from T' as above, is opt for (C, f) among trees with x, y as siblings
  - By lemma 1, some opt tree has x, y as siblings
  - Therefore, T is optimal.

$\square$

Data Compression

- Huffman is optimal.
- BUT still might do "better"
  - Huffman uses one encoding throughout a file. What if characteristics change?
  - What if data has structure? E.g. raster images, video,. . .
  - Huffman is lossless. Necessary?
- LZW, MPEG, . . .