Analysis

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Defining Efficiency

- "Runs fast on typical real problem instances"
- Pro:
  - sensible, bottom-line-oriented
- Con:
  - moving target (diff computers, compilers, Moore's law)
  - highly subjective (how fast is "fast"? what's "typical"?)

Efficiency

- Our correct TSP algorithm was incredibly slow
- Basically slow no matter what computer you have
- We want a general theory of "efficiency" that is
  - Simple
  - Objective
  - Relatively independent of changing technology
  - But still predictive - "theoretically bad" algorithms should be bad in practice and vice versa (usually)

Measuring efficiency

- Time # of instructions executed in a simple programming language
  - only simple operations (+,*,-,=,if,call,...)
  - each operation takes one time step
  - each memory access takes one time step
  - no fancy stuff (add these two matrices, copy this long string,...) built in; write it/charge for it as above
- No fixed bound on the memory size

We left out things but...

- Things we've dropped
  - memory hierarchy
    - disk, caches, registers have many orders of magnitude differences in access time
  - not all instructions take the same time in practice
  - different computers have different primitive instructions
- However,
  - the RAM model is useful for designing algorithms and measuring their efficiency
  - one can usually tune implementations so that the hierarchy etc. is not a huge factor

Complexity analysis

- Problem size n
  - Worst-case complexity: max # steps algorithm takes on any input of size n
  - Best-case complexity: min # steps algorithm takes on any input of size n
  - Average-case complexity: avg # steps algorithm takes on inputs of size n

Pros and cons:

- Best-case
  - unrealistic oversell
- Average-case
  - over what probability distribution? (different people may have different "average" problems)
  - analysis often hard
- Worst-case
  - a fast algorithm has a comforting guarantee
  - maybe too pessimistic

Why Worst-Case Analysis?

- Appropriate for time-critical applications, e.g. avionics
- Unlike Average-Case, no debate about what the right definition is
- Analysis often easier
- Result is often representative of "typical" problem instances
- Of course there are exceptions. . .

General Goals

- Characterize growth rate of (worst-case) run time as a function of problem size, up to a constant factor
- Why not try to be more precise?
  - Technological variations (computer, compiler, OS, ...) easily 10x or more
  - Being more precise is a ton of work
  - A key question is "scale up": if I can afford to do it today, how much longer will it take when my business problems are twice as large? (E.g. today: cn^2, next year: $c(2n)^2 = 4cn^2$: 4 x longer.)

Complexity

- The complexity of an algorithm associates a number T(n), the worst-case time the algorithm takes, with each problem size n.

O-notation etc

- Given two functions f and $g : N \rightarrow R$
  - $f(n)$ is $O(g(n))$ iff there is a constant $c > 0$ so that $f(n)$ is eventually always $\leq c\ g(n)$
  - $f(n)$ is $\Omega(g(n))$ iff there is a constant $c > 0$ so that $f(n)$ is eventually always $\geq c\ g(n)$
  - $f(n)$ is $\Theta(g(n))$ iff there is are constants $c_1, c_2 > 0$ so that eventually always $c_1\ g(n) \leq f(n) \leq c_2\ g(n)$

Examples

$10n^2 - 16n + 100$

- $10n^2 - 16n + 100$ is $O(n^2)$ also $O(n^3)$
  - $10n^2 - 16n + 100 \leq 11n^2$ for all $n > 10$
- $10n^2 - 16n + 100$ is $\Omega(n^2)$ also $\Omega(n)$
  - $10n^2 - 16n + 100 \geq 9n^2$ for all $n > 16$
  - Therefore also $10n^2 - 16n + 100$ is $\Theta(n^2)$
- $10n^2 - 16n + 100$ is neither $O(n)$ nor $\Omega(n^3)$

Properties

- Transitivity.
  - If $f = O(g)$ and $g = O(h)$ then $f = O(h)$.
  - If $f = \Omega(g)$ and $g = \Omega(h)$ then $f = \Omega(h)$.
  - If $f = \Theta(g)$ and $g = \Theta(h)$ then $f = \Theta(h)$.
- Additivity.
  - If $f = O(h)$ and $g = O(h)$ then $f + g = O(h)$.
  - If $f = \Omega(h)$ and $g = \Omega(h)$ then $f + g = \Omega(h)$.
  - If $f = \Theta(h)$ and $g = O(h)$ then $f + g = \Theta(h)$.

Asymptotic Bounds for Some Common Functions

- Polynomials:
  $a_0 + a_1 n + \ldots + a_d n_d$ is $\Theta(n^d)$ if $a_d > 0$
- Logarithms:
  $O(log_a n) = O(log_b n)$ for any constants $a, b > 0$
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  For all $x > 0$, $logn = O(n^x)$

“One-Way Equalities”

$2+2$ is 4 — $2n^2 + 5n$ is $O(n^3)$

$2+2=4$ — $2n2 + 5n = O(n^3)$

$4 = 2+2$ — ~~$O(n^3) = 2n^2 + 5n$~~

All dogs are mammals — ~~All mammals are dogs~~

- Bottom line:
  - OK to put big-O in R.H.S. of equality, but not left.
  - (Better, but uncommon, notation: $2n^2 + 5n \in O(n^3)$)

Working with O-Ω-Θ notation

- Claim: For any a, and any $b > 0$, $(n + a)^b$ is $\Theta(n^b)$

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(n+a)^b &\le (2n)^b && \text{for } n \ge |a| \\
&= 2^b n^b \\
&= cn^b && \text{for } c = 2^b
\end{aligned}
$$

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$$\begin{aligned} (n+a)^b &\geq (n/2)^b && \text{for } n \geq 2|a| (\text{even if } a < 0) \\ &= 2^{-b} n^b \\ &= c' n^b && \text{for } c' = 2^{-b} \end{aligned}$$

so $(n+a)^b)$ is $\Omega(n^b)$

Working with O-Ω-Θ notation (2)

- Claim: For any $a, b > 1$, $log_a n$ is $\Theta(log_b n)$

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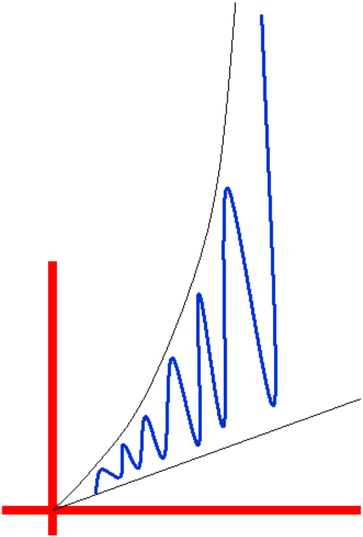
$$
\begin{aligned}
log_a b &= x \qquad a^x = b \\
a^{log_a b} &= b \\
(a^{log_a b})^{log_b n} &= b^{log_b n} \\
&= n \\
(log_a b)(log_b n) &= log_a n \\
c log_b n &= log_a n \quad \text{for the constant } c = log_a b
\end{aligned}
$$

So, $log_b n = \Theta(log_a n) = \Theta(\log n)$

Big-Theta, etc. not always "nice"

$$f(x) = \begin{cases} n^2 & \text{if } n \text{ even,} \\ n & \text{else} \end{cases}$$

- $f(n) \neq \Theta(n^a)$ for any a.
- Fortunately, this is rare

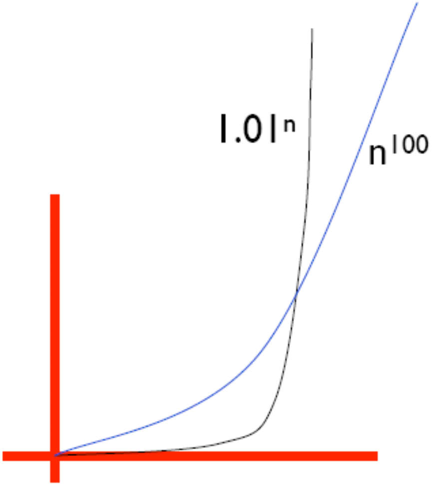


A Possible Misunderstanding?

- We have looked at
  - type of complexity analysis
    - worst-, best-, average-case
  - types of function bounds
    - O, Ω, Θ
- These two considerations are independent of each other
  - one can do any type of function bound with any type of complexity analysis - measuring different things with same yardstick

Asymptotic Bounds for Some Common Functions

- Exponentials. For all $r > 1$ and all $d > 0$, $n^d = O(r^n)$.



Polynomial time

- Running time is $O(n^d)$ for some constant d independent of the input size n.

Why It Matters

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time.

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

Example: Studying Protein Interactions

- Yeast: $\approx 6k$ proteins
  - $n^2 \longrightarrow$ 36 sec
  - $n^3 \longrightarrow$ 2.5 days
  - $n^4 \longrightarrow$ 41 years. But maybe could split it over 100 computers ...
  - $2^n \longrightarrow > 10^{1000}$ years
  - $1.01^n \longrightarrow 10^{10}$ years
- Human: $\approx 25k$ proteins
  - $n^2 \longrightarrow$ 10 min
  - $n^3 \longrightarrow$ 0.5 years
  - $n^4 \longrightarrow 10^5$ years

Key Facts

- $\log n \leq n^d \leq d^n$ (asymptotically)
- Polynomial time (or faster) is good
- Exponential time is bad

Geek-speak Faux Pas du Jour

- “Any comparison-based sorting algorithm requires at least O(n log n) comparisons.”
    - Statement doesn’t ”type-check.”
    - Use Ω for lower bounds.

Summary

- Typical initial goal for algorithm analysis is to find a
  reasonably tight (i.e., Θ if possible)
  asymptotic (i.e., O or Θ)
  bound on (usually upper bound)
  worst case running time
  as a function of problem size
- This is rarely the last word, but often helps separate good algorithms from blatantly poor ones - so you can concentrate on the good ones!