

# Stable Matching

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# Lecture Outline

## 1 Administration

# Lecture Outline

1 Administration

2 Stable Matching

# Admin

- Subscribe to the Mailing List!
- Textbook on reserve
- Office Hours
- Homework 1 released soon

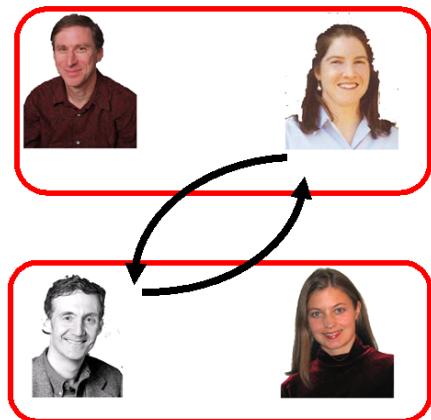
# Prom Dates

- Hank asks Ruth to the school Prom
- Ruth says yes ... but would rather go with Ed
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- Ruth and Ed would prefer to be going with each other!



# Abstract Definition of Stable Matching

- Input:
  - Two sets, eg. men  $M = \{m_1, \dots, m_n\}$ , and women  $W = \{w_1, \dots, w_n\}$
  - Preferences: Each man ranks all of the women (and vice versa)
- Desired Output: Perfect, Stable Matching  $S$ 
  - Matching: set  $S$  of pairs  $(m, w)$
  - Each person appears exactly once
  - No **instabilities** in  $S$

# Instabilities

Our first matching was bad because of an **instability**

- $(m, w), (m', w') \in S$
- $m$  prefers  $w'$  over  $w$
- $w'$  prefers  $m$  over  $m'$



# The Gale-Shapley Algorithm

- 1 Initially, nobody is married
- 2 An unmarried man  $m$  chooses the highest ranked woman  $w$  that he has not yet proposed to.
  - If  $w$  is not engaged, then  $m$  and  $w$  become engaged
  - If  $w$  is engaged to another man  $m'$ , if  $w$  prefers  $m$  over  $m'$ , then  $m$  and  $w$  become engaged.
  - Otherwise,  $w$  rejects  $m$  proposal, and is still engaged to  $m'$ .
- 3 Repeat 2 until no one is free
- 4 All engagements are final

# How do we know it works?

- 1 Does it terminate?
- 2 Does it give the right answer?
- 3 How long does it take?

# Termination of Gale-Shapley

- Useful to have a measure of **progress**
- We will use the number of proposals
- Each man will not propose more than once to each women.
- So there are only  $n^2$  possible proposals

# Correctness of Gale-Shapley: Everyone married

At any point during the G-S algorithm, if a man  $m$  is free, then he has not yet proposed to some woman

**Proof.**

By Contradiction

Suppose  $m$  is free, but has already proposed to every woman. Then every woman must be engaged. But there are  $n$  men,  $n$  women, so there must be  $n$  engaged men. Then  $m$  cannot be free. □

# Women's engagements improve

## Lemma

*After the first proposal, women are always engaged. The sequence of engagements only gets better (from the women's point of view).*

# Correctness of Gale-Shapley: No Instabilities

If G-S algo returns a matching  $S$ , then  $S$  is stable.

Proof.

By Contradiction

- Suppose there is an instability: two pairs  $(m, w)$  and  $(m', w')$ , such that
  - $m$  prefers  $w'$  over  $w$
  - $w'$  prefers  $m$  over  $m'$
- We know  $m$ 's last proposal was to  $w$ . If  $m$  did not already propose to  $w'$ , then  $m$  prefers  $w$  to  $w'$ , a contradiction.

# Correctness of Gale-Shapley: No Instabilities (2)

## Proof.

### Continued

- So then  $m$  proposed to  $w'$ . But  $w'$  must have rejected  $m$ , in favor of some other man,  $m''$ .
- We know that  $w'$  ends paired with some one that she likes at least as much as  $m''$ . So she liked  $m'$  at least as much as  $m''$ .
- But this contradicts our basic assumption, that  $w'$  prefers  $m$  over  $m'$ .



# General Lessons

- Proof by **Contradiction**: Try to prove yourself wrong.
- Proof gives us a **guarantee** — heuristic ideas don't give that to us
- Measure of **Progress**



# A bad “algorithm”

- 1 Order the men & women

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- 1 Order the men & women
- 2 Each man proposes to highest available women
- 3 If there is an instability, each woman proposes to highest available man
- 4 Keep repeating as long as there is an instability

# No progress

$m_1$	$w_3$	$w_2$	$w_1$
$m_2$	$w_1$	$w_2$	$w_3$
$m_3$	$w_1$	$w_2$	$w_3$

$w_1$	$m_3$	$m_2$	$m_1$
$w_2$	$m_3$	$m_2$	$m_1$
$w_3$	$m_1$	$m_2$	$m_3$

What happens after the men chose once, and then the women chose once?

# More Details in the book

- What if women propose? (better for the women!)
- Does order matter?

# From abstract to concrete

- We started with prom / marriage, went to abstract model ...
- can we go from abstract model to concrete?
  - Scheduling non-conference games (ACC/BigTen challenge)
  - School admissions — a little different though (your homework)



# Homework Advice

- Start early! Think about problems in the back of your brain ...
- Use a study group for brainstorming (but write answers on your own)
- Homework worth a lot in this course

