Stable Matching

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Lecture Outline

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Admin

- Subscribe to the Mailing List!
- Textbook on reserve
- Office Hours
- Homework 1 released soon

Prom Dates

- Hank asks Ruth to the school Prom
- Ruth says yes ... but would rather go with Ed
- Since Ruth is taken, Ed asks Crystal (but really, he'd rather go with Ruth)

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- Ruth and Ed would prefer to be going with each other!

Abstract Defintion of Stable Matching

- Input:
  - Two sets, eg. men $M = \{m_1, \ldots, m_n\}$, and women $W = \{w_1, \ldots, w_n\}$
  - Preferences: Each man ranks all of the women (and vice versa)
- Desired Output: Perfect, Stable Matching S
  - Matching: set S of pairs (m, w)
  - Each person appears exactly once
  - No instabilities in S

Instabilities

Our first matching was bad because of an instability

- $(m, w), (m', w') \in S$
- m prefers w' over w
- w' prefers m over m'

The Gale-Shapley Algorithm

1. Initially, nobody is married
2. An unmarried man m chooses the highest ranked woman w that he has not yet proposed to.
   - If w is not engaged, then m and w become engaged
   - If w is engaged to another man m', if w prefers m over m', then m and w become engaged.
   - Otherwise, w rejects m proposal, and is still engaged to m'.
3. Repeat 2 until no one is free
4. All engagements are final

How do we know it works?

1. Does it terminate?
2. Does it give the right answer?
3. How long does it take?

Termination of Gale-Shapley

- Useful to have a measure of progress
- We will use the number of proposals
- Each man will not propose more than once to each women.
- So there are only n^2 possible proposals

Correctness of Gale-Shapley: Everyone married

At any point during the G-S algorith, if a man m is free, then he has not yet proposed to some woman

Proof.

By Contradiction

Suppose m is free, but has already proposed to every woman. Then every woman must be engaged. But there are n men, n women, so there must be n engaged men. Then m cannot be free. □

Women's engagements improve

Lemma

After the first proposal, women are always engaged. The sequence of engagements only gets better(from the women's point of view).

Correctness of Gale-Shapley: No Instabilities

If G-S algo returns a matching S, then S is stable.

Proof.

By Contradiction

- Suppose there is an instability: two pairs (m, w) and (m', w'), such that
  - m prefers w' over w'
  - w' prefers m over m'
- We know m's last proposal was to w. If m did not already propose to w', then m prefers w to w', a contradiction.

Correctness of Gale-Shapley: No Instabilities (2)

Proof.

Continued

- So then m propsed to w'. But w' must have rejected m , in favor of some other man, m''.
- We know that w' ends paired with some one that she likes at least as much as m''. So she liked m' at least as much as m''.
- But this contradicts our basic assumption, that w' prefers m over m'.

General Lessons

- Proof by Contradiction: Try to prove yourself wrong.
- Proof gives us a guarantee — heuristic ideas don't give that to us
- Measure of Progress

A bad “algorithm”

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A bad "algorithm"

1. Order the men & women
2. Each man proposes to highest available women
3. If there is an instability, each woman proposes to highest available man
4. Keep repeating as long as there is an instability

No progress

m_1	w_3	w_2	w_1
m_2	w_1	w_2	w_3
m_3	w_1	w_2	w_3

w_1	m_3	m_2	m_1
w_2	m_3	m_2	m_1
w_3	m_1	m_2	m_3

What happens after the men chose once, and then the women chose once?

More Details in the book

- What if women propose? (better for the women!)
- Does order matter?

From abstract to concrete

- We started with prom / marriage, went to abstract model ...
- can we go from abstract model to concrete?
  - Scheduling non-conference games (ACC/BigTen challenge)
  - School admissions — a little different though (your homework)

Homework Advice

- Start early! Think about problems in the back of your brain ...
- Use a study group for brainstorming (but write answers on your own)
- Homework worth a lot in this course