CSE 417: Algorithms and Computational Complexity

2: Analysis

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Efficiency

- Our correct TSP algorithm was incredibly slow
- Basically slow no matter what computer you have
- We would like a general theory of "efficiency" that is
  - Simple
  - Relatively independent of changing technology
  - But still useful for prediction - "theoretically bad" algorithms should be bad in practice and vice versa (usually)

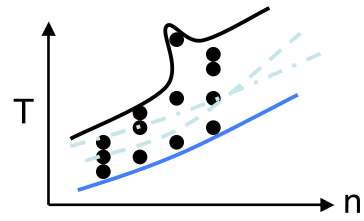
Measuring efficiency: The RAM model

- RAM = Random Access Machine
- Time ≈ # of instructions executed in an ideal assembly language
  - each simple operation (+,*,-,=,if,call) takes one time step
  - each memory access takes one time step
- No bound on the memory size

We left out things but...

- Things we've dropped
  - memory hierarchy
    - disk, caches, registers have many orders of magnitude differences in access time
  - not all instructions take the same time in practice
- However,
  - the RAM model is useful for designing algorithms and measuring their efficiency
  - one can usually tune implementations so that the hierarchy etc. is not a huge factor

Complexity analysis



- Problem size n
  - **Worst-case complexity**: **max** # steps algorithm takes on any input of size n
  - **Best-case complexity:** **min** # steps algorithm takes on any input of size n
  - **Average-case complexity**: **avg** # steps algorithm takes on inputs of size n

Pros and cons:

- Best-case
  - unrealistic overselling
  - can "cheat": tune algorithm for one easy input
- Worst-case
  - a fast algorithm has a comforting guarantee
  - no way to cheat by hard-coding special cases
  - maybe too pessimistic
- Average-case
  - over what probability distribution? (different people may have different "average" problems)
  - analysis hard

Why Worst-Case Analysis?

- Appropriate for time-critical applications, e.g. avionics
- Unlike Average-Case, no debate about what the right definition is
- Analysis often easier
- Result is often representative of "typical" problem instances
- Of course there are exceptions…

General Goals

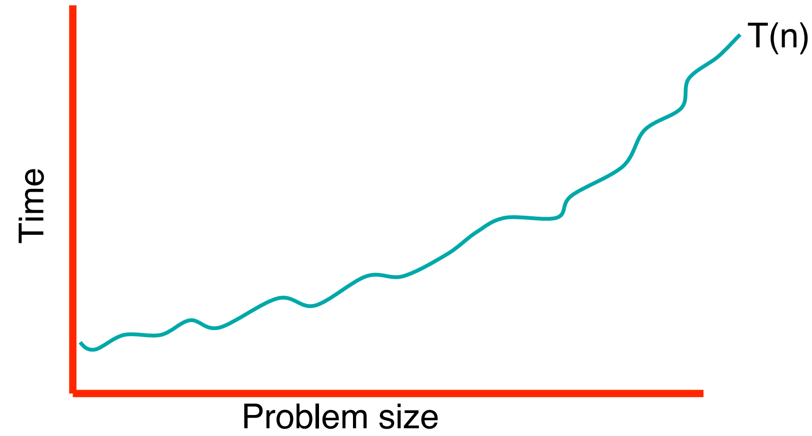
- Characterize *growth rate* of run time as a function of problem size, up to a *constant factor*
- Why not try to be more precise?
  - Technological variations (computer, compiler, OS, …) easily 10x or more
  - Being more precise is a ton of work
  - A key question is "scale up": if I can afford to do it today, how much longer will it take when my business problems are twice as large? (E.g. today: cn^2, next year: $c(2n)^2 = 4cn^2$: 4 x longer.)

Complexity

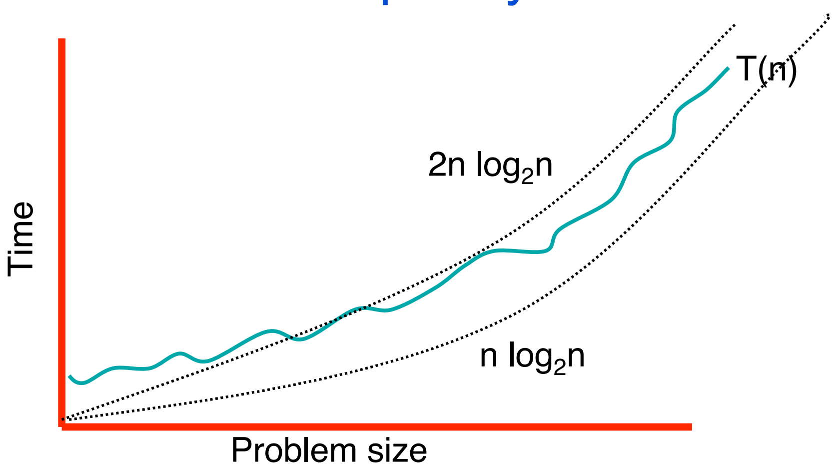
- The complexity of an algorithm associates a number **T(n)**, the best/worst/average-case time the algorithm takes, with each problem size **n**.

- Mathematically,
  - **T: $N^+ \rightarrow R^+$**
  - that is **T** is a function that maps positive integers giving problem size to positive real numbers giving number of steps.

Complexity



Complexity



O-notation etc

- Given two functions **f** and **g:N→R**
  - **f(n)** is **O(g(n))** iff there is a constant **c**>0 so that **f(n)** is eventually always $\leq$ **c g(n)**
  - **f(n)** is **Ω(g(n))** iff there is a constant **c**>0 so that **f(n)** is eventually always $\geq$ **c g(n)**
  - **f(n)** is **Θ(g(n))** iff there is are constants c_1 and c_2>0 so that eventually always $c_1 g(n) \leq f(n) \leq c_2 g(n)$

Examples

- **$10n^2$-16n+100** is **$O(n^2)$** also **$O(n^3)$**
  - $10n^2 - 16n + 100 \le 11n^2$ for all $n \ge 10$
- **$10n^2$-16n+100** is **$\Omega(n^2)$** also **$\Omega(n)$**
  - $10n^2 - 16n + 100 \ge 9n^2$ for all $n \ge 16$
  - Therefore also **$10n^2$-16n+100** is **$\Theta(n^2)$**
- **$10n^2$-16n+100** is **not $O(n)$** also **not $\Omega(n^3)$**

Properties

- Transitivity.
  - If $f = O(g)$ and $g = O(h)$ then $f = O(h)$.
  - If $f = \Omega(g)$ and $g = \Omega(h)$ then $f = \Omega(h)$.
  - If $f = \Theta(g)$ and $g = \Theta(h)$ then $f = \Theta(h)$.
- Additivity.
  - If $f = O(h)$ and $g = O(h)$ then $f + g = O(h)$.
  - If $f = \Omega(h)$ and $g = \Omega(h)$ then $f + g = \Omega(h)$.
  - If $f = \Theta(h)$ and $g = O(h)$ then $f + g = \Theta(h)$.

Asymptotic Bounds for Some Common Functions

- Polynomials:
  $a_0 + a_1 n + \ldots + a_d n^d$ is $\Theta(n^d)$ if $a_d > 0$
- Logarithms:
  $O(\log_a n) = O(\log_b n)$ for any constants a,b > 0
- Logarithms:
  For all $x > 0$, $\log n = O(n^x)$

log grows slower than every polynomial

"One-Way Equalities"

- "2 + 2 is 4" vs 2 + 2 = 4 vs 4 = 2 + 2
- "Every dog is a mammal" vs
  "Every mammal is a dog"
- $2n^2 + 5n$ is $O(n^3)$ vs
  $2n^2 + 5n = O(n^3)$ vs
  $O(n^3) = 2n^2 + 5n$

  FALSE
- OK to put big-O in R.H.S. of equality, but not left. Better notation: $T(n) \in O(f(n))$.

Working with O-Ω-Θ notation

Claim: For any a, and any b>0, $(n+a)^b$ is $\Theta(n^b)$

- $(n+a)^b \le (2n)^b$ for $n \ge |a|$
  $= 2^b n^b$
  $= cn^b$ for $c = 2^b$

  so $(n+a)^b$ is $O(n^b)$

- $(n+a)^b \ge (n/2)^b$ for $n \ge 2|a|$ (even if a <0)
  $= 2^{-b} n^b$
  $= c'n$ for $c' = 2^{-b}$

  so $(n+a)^b$ is $\Omega(n^b)$

Working with O-Ω-Θ notation

Claim: For any a, b>1 $\log_a n$ is $\Theta(\log_b n)$

$$\log_a b = x \text{ means } a^x = b$$
$$a^{\log_a b} = b$$
$$(a^{\log_a b})^{\log_b n} = b^{\log_b n} = n$$
$$(\log_a b)(\log_b n) = \log_a n$$
$$c \log_b n = \log_a n \text{ for the constant } c = \log_a b$$
So:
$$\log_b n = \Theta(\log_a n) = \Theta(\log n)$$

Domination

- **f(n)** is **o(g(n))** iff **$\lim_{n\to\infty}$ f(n)/g(n)=0**
  - that is **g(n) dominates f(n)**
- If $\alpha \le \beta$ then n^α is $O(n^\beta)$
- If $\alpha < \beta$ then n^α is $o(n^\beta)$
- **Note:**
  if **f(n)** is **Θ(g(n))** then it cannot be **o(g(n))**

Working with little-o

- $n^2 = o(n^3)$ [Use algebra]:

$$\lim_{n\to\infty} \frac{n^2}{n^3} = \lim_{n\to\infty} \frac{1}{n} = 0$$

- $n^3 = o(e^n)$ [Use L'Hospital's rule 3 times]:

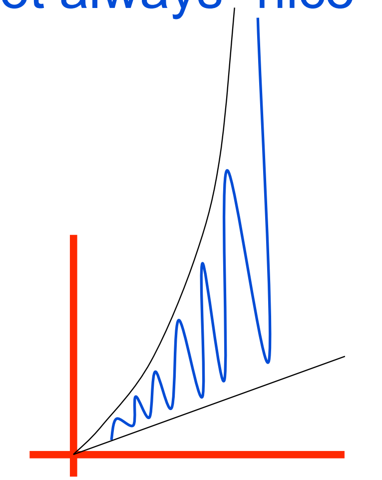
$$\lim_{n\to\infty} \frac{n^3}{e^n} = \lim_{n\to\infty} \frac{3n^2}{e^n} = \lim_{n\to\infty} \frac{6n}{e^n} = \lim_{n\to\infty} \frac{6}{e^n} = 0$$

Big-Theta, etc. not always "nice"

$$f(n) = \begin{cases} n^2, & n \text{ even} \\ n, & n \text{ odd} \end{cases}$$

f(n) $\neq \Theta(n^a)$ for any *a*.

Fortunately, such nasty cases are rare



f(n log n) $\neq \Theta(n^a)$ for any *a, either, but at least it's simpler.*

A Possible Misunderstanding?

- We have looked at
  - type of complexity analysis
    - worst-, best-, average-case
  - types of function bounds
    - O, Ω, Θ
- These two considerations are independent of each other
  - one can do any type of function bound with any type of complexity analysis

Insertion Sort:

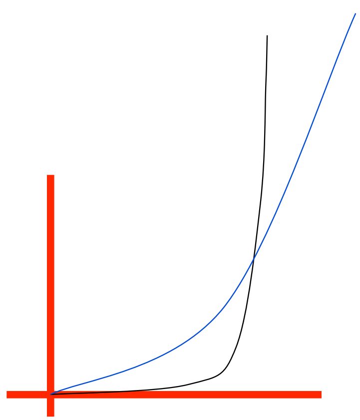
$\Omega(n^2)$ (worst case)

O(n) (best case)

Asymptotic Bounds for Some Common Functions

- Exponentials. For all r > 1 and all d > 0, $n^d = O(r^n)$.

every exponential grows faster than every polynomial



Polynomial time

- Running time is $O(n^d)$ for some constant d independent of the input size n.

Why It Matters

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time.

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

Geek-speak Faux Pas du Jour

- "Any comparison-based sorting algorithm requires at least O(n log n) comparisons."
  - Statement doesn't "type-check."
  - Use Ω for lower bounds.