CSE 417: Algorithms and Computational Complexity

Winter 2005

Graphs and Graph Algorithms

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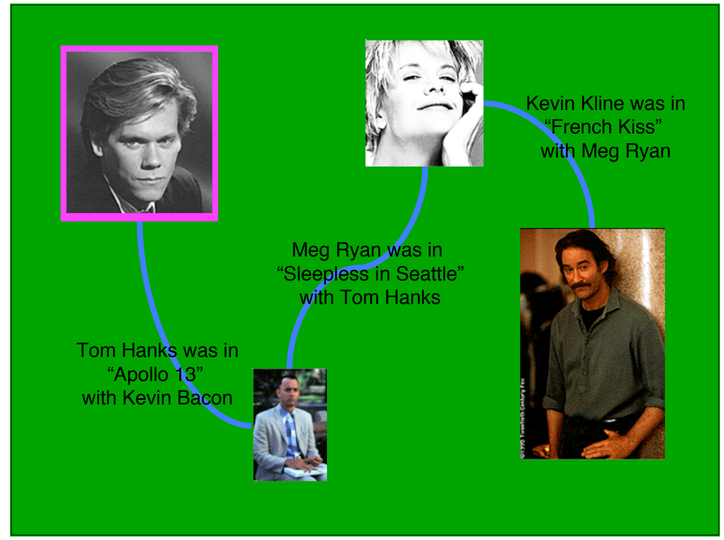

Objects & Relationships

- The Kevin Bacon Game:
  - Actors
  - Two are related if they've been in a movie together
- Exam Scheduling:
  - Classes
  - Two are related if they have students in common
- Traveling Salesperson Problem:
  - Cities
  - Two are related if can travel directly between them

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Graphs

- An extremely important formalism for representing (binary) relationships
- Objects: "vertices", aka "nodes"
- Relationships between pairs: "edges", aka "arcs"
- Formally, a graph G = (V, E) is a pair of sets, V the vertices and E the edges

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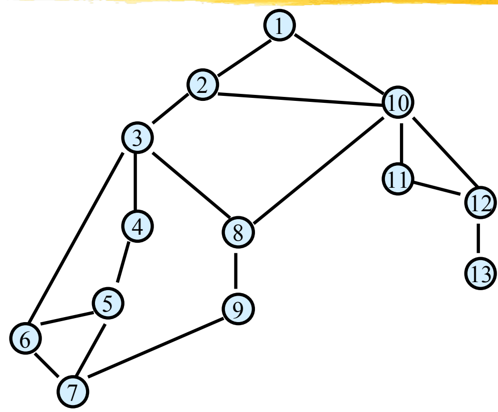
Undirected Graph G = (V,E)



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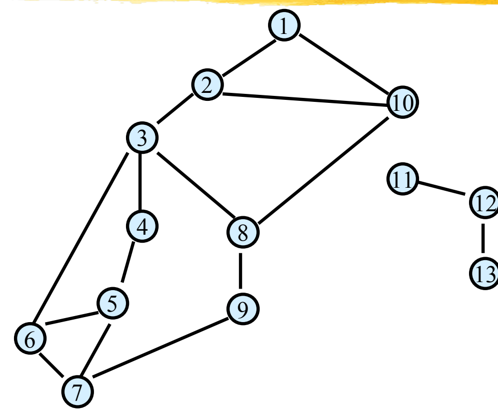
Undirected Graph G = (V,E)



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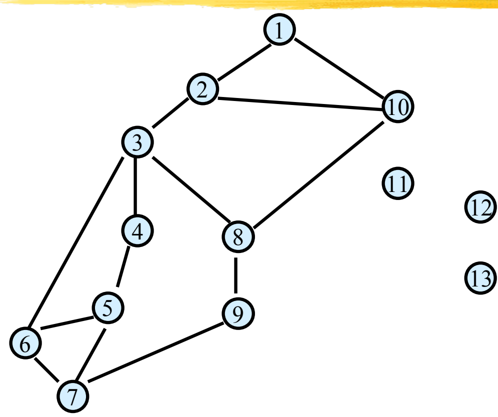
Undirected Graph G = (V,E)



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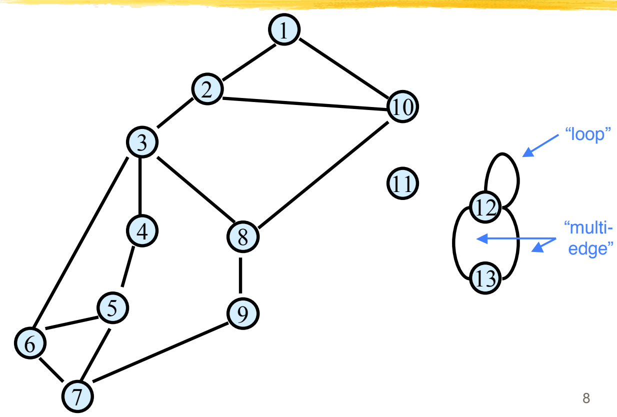
Undirected Graph G = (V,E)



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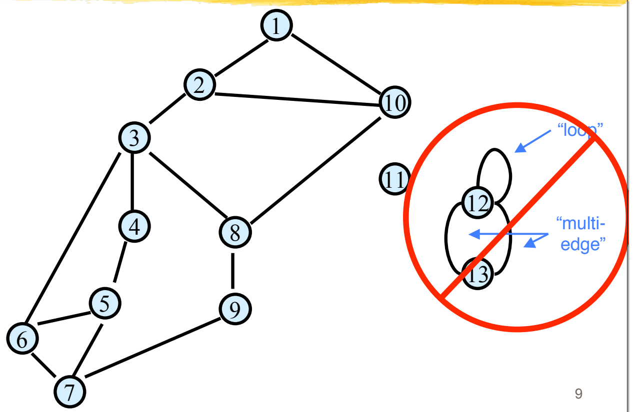
Undirected Graph G = (V,E)



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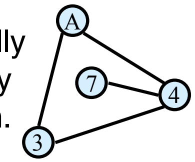
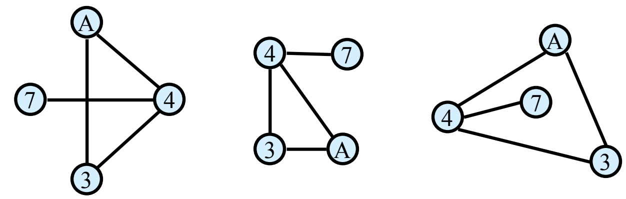
Graphs don’t live in Flatland

- Geometrical drawing is mentally convenient, but mathematically irrelevant: 4 drawings, 1 graph.





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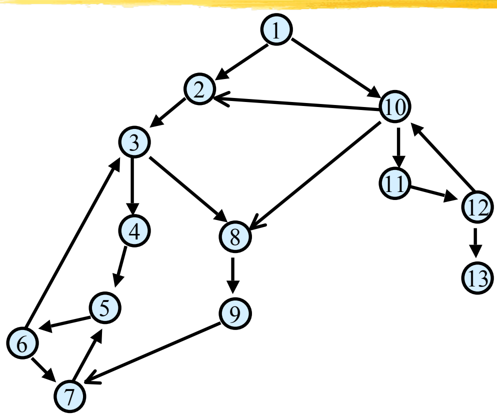
Directed Graph G = (V,E)



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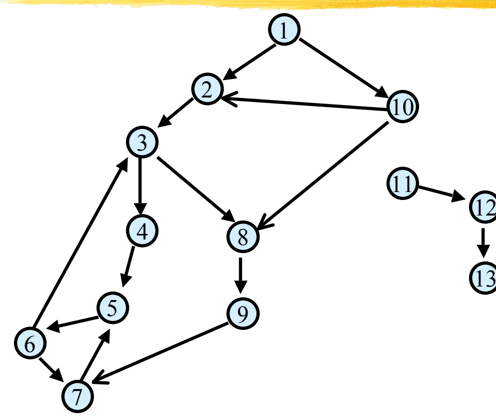
Directed Graph G = (V,E)



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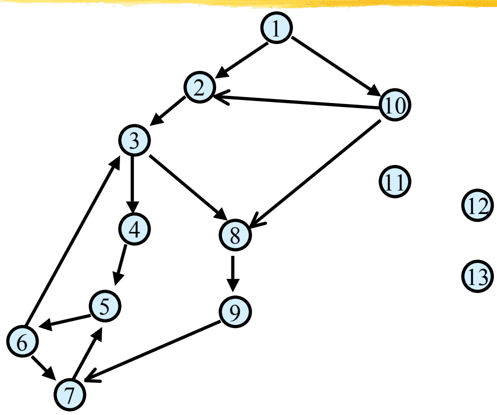
Directed Graph G = (V,E)



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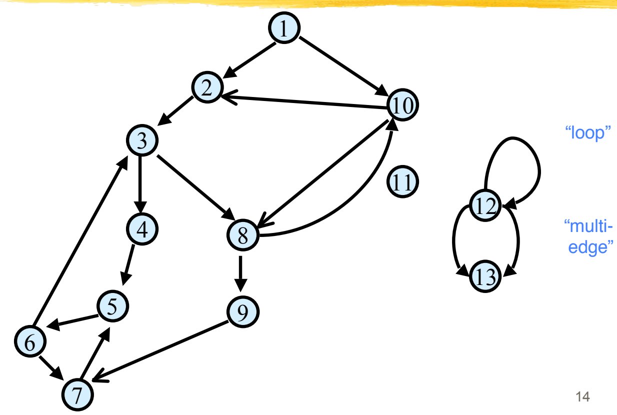
Directed Graph G = (V,E)



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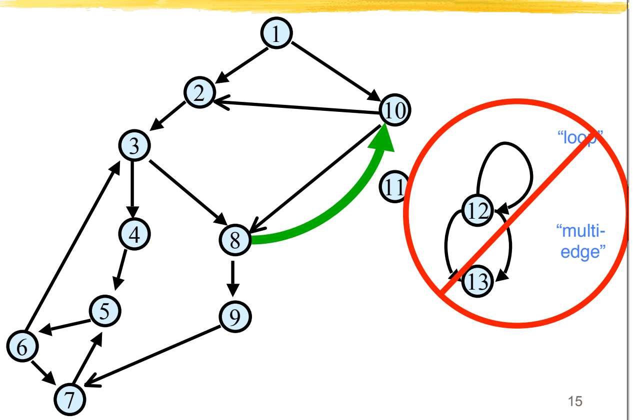
Directed Graph G = (V,E)



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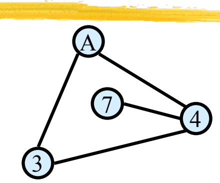
Specifying undirected graphs as input

- What are the vertices?
  - Explicitly list them: {“A”, “7”, “3”, “4”}
- What are the edges?
  - Either, set of edges {{A,3}, {7,4}, {4,3}, {4,A}}
  - Or, (symmetric) adjacency matrix:



	A	7	3	4
A	0	0	1	1
7	0	0	0	1
3	1	0	0	1
4	1	1	1	0

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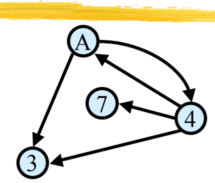
Specifying directed graphs as input

- What are the vertices
  - Explicitly list them: {“A”, “7”, “3”, “4”}
- What are the edges
  - Either, set of directed edges: {(A,4), (4,7), (4,3), (4,A), (A,3)}
  - Or, (nonsymmetric) adjacency matrix:



	A	7	3	4
A	0	0	1	1
7	0	0	0	0
3	0	0	0	0
4	1	1	1	0

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Vertices vs # Edges

- Let G be an undirected graph with n vertices and m edges
- How are n and m related?
- Since
  - every edge connects two *different* vertices (no loops), and
  - no two edges connect the *same* two vertices (no multi-edges),

  it must be true that: $0 \le m \le n(n-1)/2 = O(n^2)$

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More Cool Graph Lingo

- A graph is called *sparse* if $m \ll n^2$, otherwise it is *dense*
  - Boundary is somewhat fuzzy; O(n) edges is certainly sparse, $\Omega(n^2)$ edges is dense.
- Sparse graphs are common in practice
  - E.g., all planar graphs are sparse
- Q: which is a better run time, O(n+m) or $O(n^2)$?

A: $O(n+m) = O(n^2)$, but n+m usually way better!

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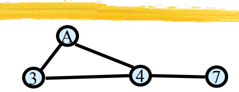
Representing Graph G = (V,E) n vertices, m edges

- Vertex set $V = \{v_1, \ldots, v_n\}$
- Adjacency Matrix A
  - $A[i,j] = 1$ iff $(v_i,v_j) \in E$
  - Space is n^2 bits



	A	7	3	4
A	0	0	1	1
7	0	0	0	1
3	1	0	0	1
4	1	1	1	0

- Advantages:
  - O(1) test for presence or absence of edges.
  - compact if in packed binary form for large m
- Disadvantages: inefficient for sparse graphs

$m \ll n^2$

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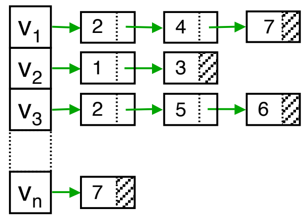
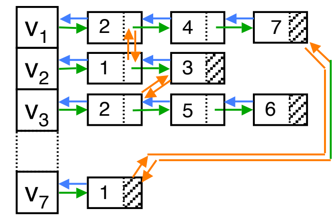
Representing Graph G=(V,E)
n vertices, m edges

- Adjacency List:
  - O(n+m) words
- Advantages:
  - Compact for sparse graphs
  - Easily see all edges
- Disadvantages
  - More complex data structure
  - no O(1) edge test



Representing Graph G=(V,E)
n vertices, m edges

- Adjacency List:
  - O(n+m) words



- Back- and cross pointers more work to build, but allow easier traversal and deletion of edges, *if needed,* (don't bother if not)

Graph Traversal

- Learn the basic structure of a graph
- "Walk," <u>via edges</u>, from a fixed starting vertex v to all vertices reachable from v
- Three states of vertices
  - **undiscovered**
  - **discovered**
  - **fully-explored**

Breadth-First Search

- Completely explore the vertices in order of their distance from v
- Naturally implemented using a queue

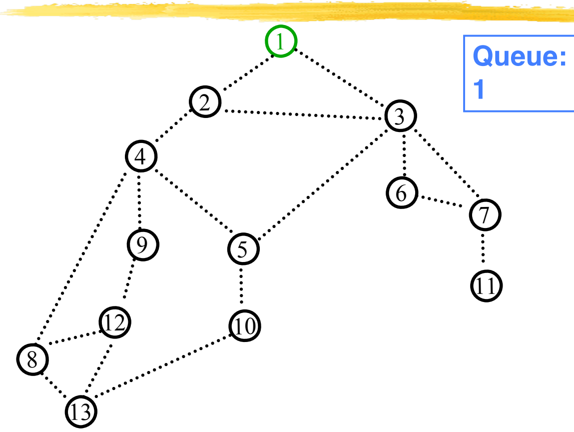
BFS(v)

Global initialization: mark all vertices "undiscovered"

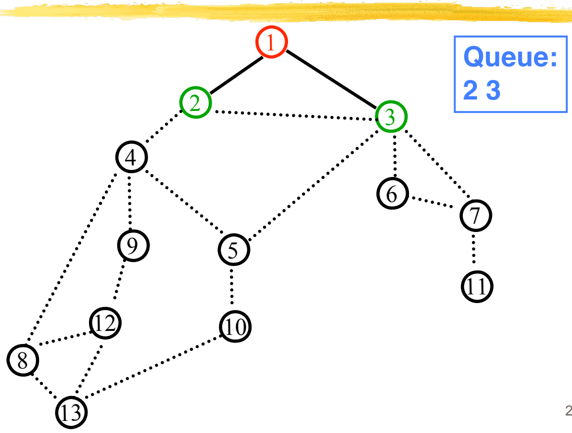
```
BFS(v)
    mark  v "discovered"
    queue = v
    while queue not empty
        u = remove_first(queue)
        for each edge {u,x}
            if (x is undiscovered)
                mark x discovered
                append x on queue
        mark u completed
```

Exercise: modify code to number vertices & compute level numbers

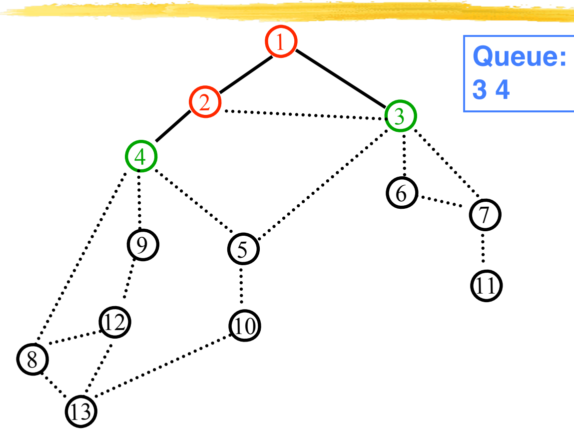
BFS(v)

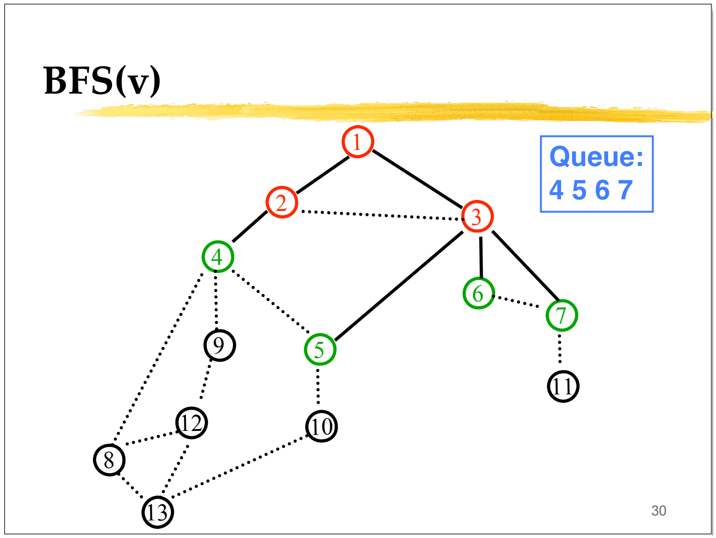


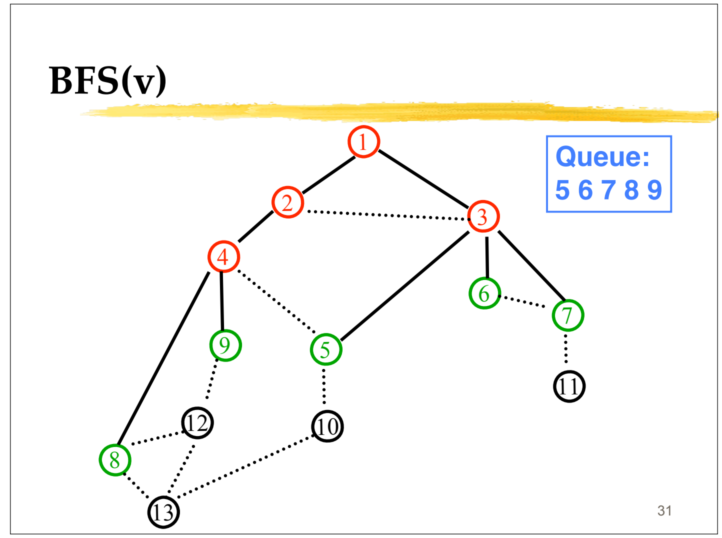
BFS(v)

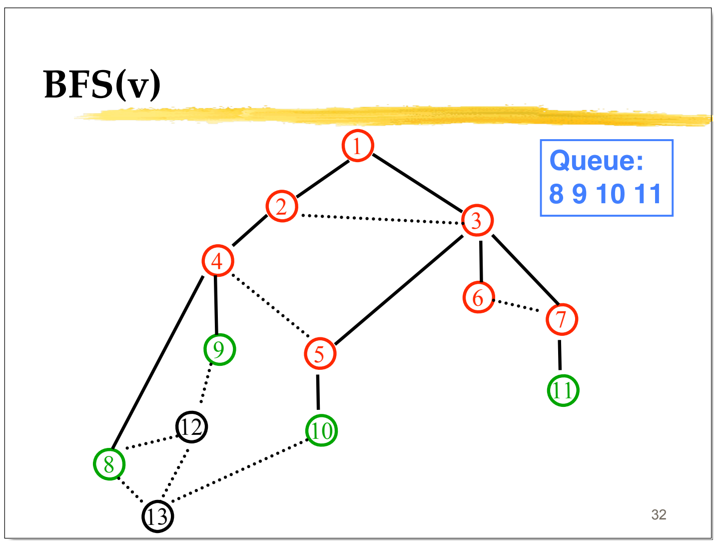


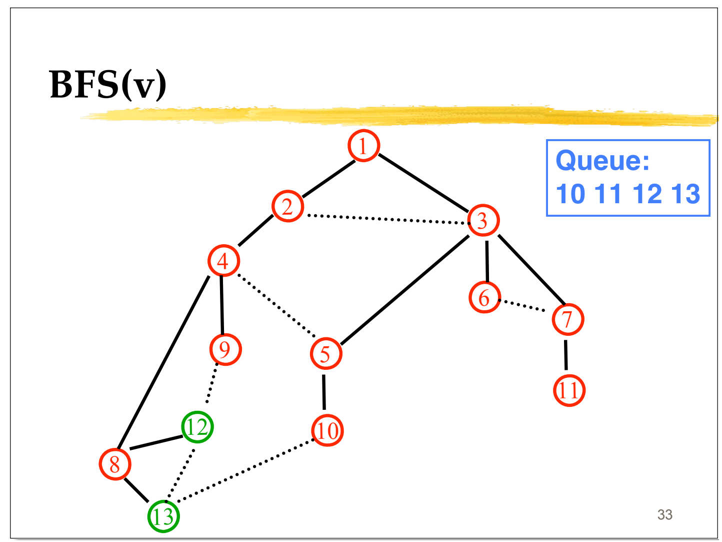
BFS(v)



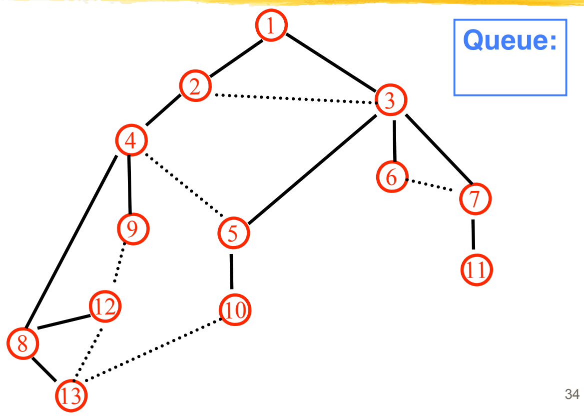

BFS(v)
Queue:
4 5 6 7
30


BFS(v)
Queue:
5 6 7 8 9
31


BFS(v)
Queue:
8 9 10 11
32


BFS(v)
Queue:
10 11 12 13
33

BFS(v)



BFS analysis

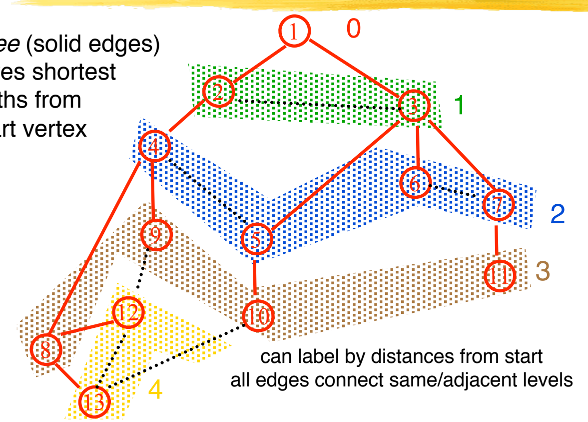
- Each edge is explored once from each end-point (at most)
- Each vertex is discovered by following a different edge
- Total cost O(m) where m=# of edges

Properties of (Undirected) BFS(v)

- BFS(v) visits x if and only if there is a path in G from v to x.
- Edges into then-undiscovered vertices define a ***tree*** – the "breadth first spanning tree" of G
- Level i in this tree are exactly those vertices u such that the shortest path (in G, not just the tree) from the root v is of length i.
- ***All*** non-tree edges join vertices on the same or adjacent levels

BFS Application: Shortest Paths

Tree (solid edges) gives shortest paths from start vertex



can label by distances from start
all edges connect same/adjacent levels

Why fuss about trees?

- Trees are simpler than graphs
- Ditto for algorithms on trees vs on graphs
- So, this is often a good way to approach a graph problem: find a "nice" tree in the graph, i.e., one such that non-tree edges have some simplifying structure
- E.g., BFS finds a tree s.t. level-jumps are minimized
- DFS (next) finds a different tree, but it also has interesting structure…

Graph Search Application: Connected Components

- Want to answer questions of the form:
  - given vertices u and v, is there a path from u to v?
- Idea: create array A such that
  A[u] = smallest numbered vertex that is connected to u
- question reduces to whether A[u]=A[v]?

Q: Why not create 2-d array Path[u,v]?

Graph Search Application: Connected Components

- initial state: all v undiscovered
  for v=1 **to** n **do**
    if state(v) != fully-explored **then**
      BFS(v): **setting** A[u] ←v **for each** u **found**
      (and marking u discovered/fully-explored)
    endif
  endfor
- Total cost: O(n+m)
  - each edge is touched a constant number of times
  - works also with DFS

Depth-First Search

- Follow the first path you find as far as you can go
- Back up to last unexplored edge when you reach a dead end, then go as far you can
- Naturally implemented using recursive calls or a stack

DFS(v) - explicit stack

Exercise: modify to compute vertex numbering

Global Initialization: mark all vertices "undiscovered"

DFS(v)
  mark v "discovered"
  push (v,1) onto empty stack
  while stack not empty
    (u,i) = pop(stack)
    for (; i ≤ # of neighbors of u; i++)
      x = i^{th} edge on u's edge list
      if (x is undiscovered)
        mark x "discovered"
        push (u,i+1)   // save info to resume with u's next edge,
        u = x   // after exploring from x,
        i = 1   // (starting with its first edge)
    mark u completed

Idea: stack of unfinished vertices, plus pointers into their edge lists to say what work remains to finish.

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DFS(v) – Recursive version

Global Initialization:
  mark all vertices v "undiscovered" via v.dfs# = -1
  dfscounter = 0

DFS(v)
  v.dfs# = dfscounter++   // mark v "discovered", & number it
  for each edge (v,x)
    if (x.dfs# = -1)   // tree edge (x previously undiscovered)
      DFS(x)
    else …   // code for back-, fwd-, parent,
        // edges, if needed
  // mark v "completed," if needed

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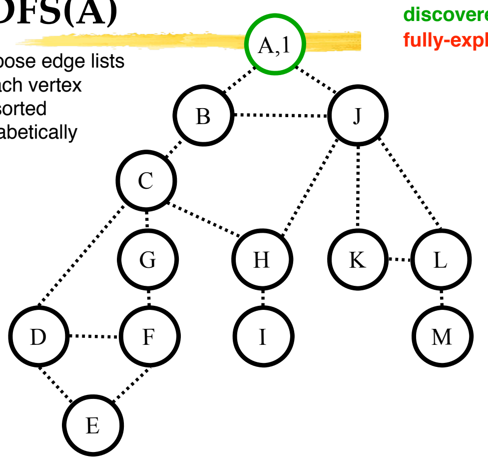
DFS(A)

Color code:
undiscovered
discovered
fully-explored

Suppose edge lists at each vertex are sorted alphabetically



Call Stack (Edge list):
A (B,J)

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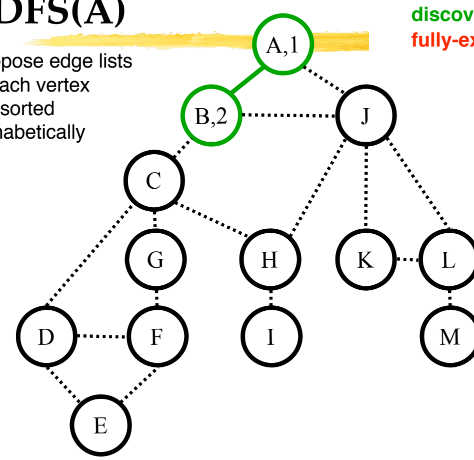
DFS(A)

Color code:
undiscovered
discovered
fully-explored

Suppose edge lists at each vertex are sorted alphabetically



Call Stack: (Edge list)
A (B,J)
B (A,C,J)

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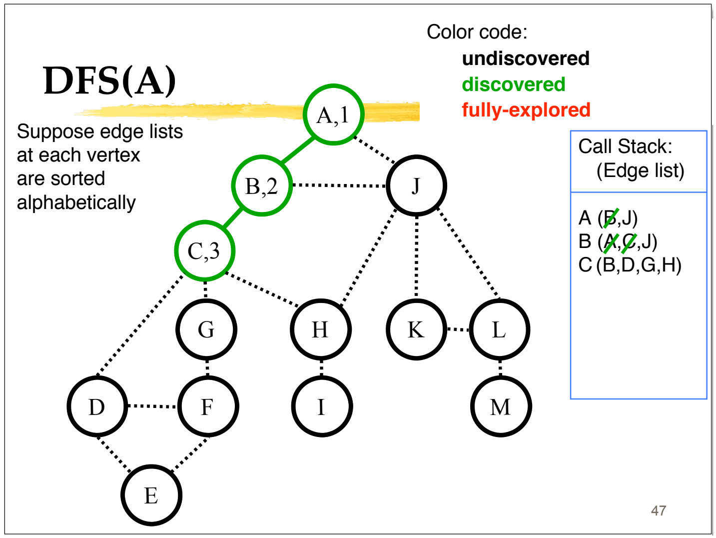
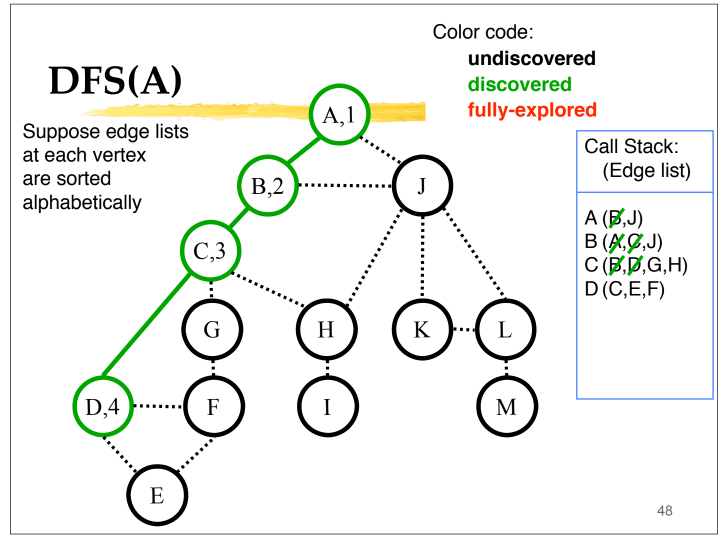
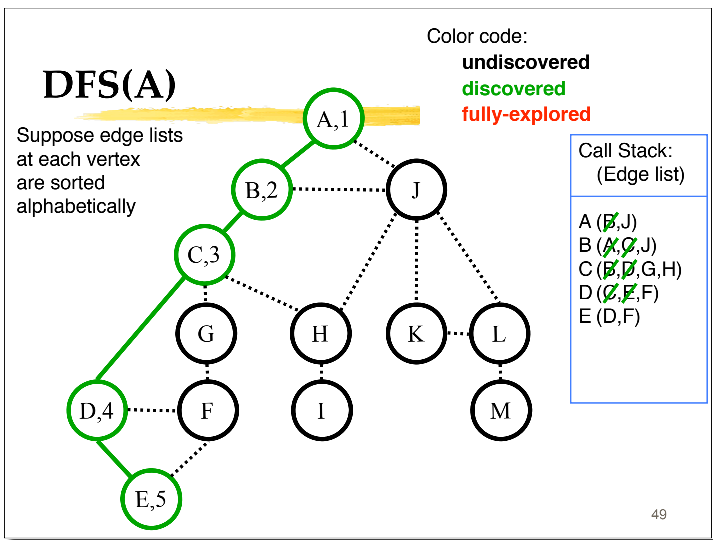

DFS(A)
Color code:
undiscovered
discovered
fully-explored
Suppose edge lists at each vertex are sorted alphabetically
A,1
B,2
C,3
D
E
F
G
H
I
J
K
L
M
Call Stack: (Edge list)
A (B,J)
B (A,C,J)
C (B,D,G,H)
47


DFS(A)
Color code:
undiscovered
discovered
fully-explored
Suppose edge lists at each vertex are sorted alphabetically
A,1
B,2
C,3
D,4
E
F
G
H
I
J
K
L
M
Call Stack: (Edge list)
A (B,J)
B (A,C,J)
C (B,D,G,H)
D (C,E,F)
48


DFS(A)
Color code:
undiscovered
discovered
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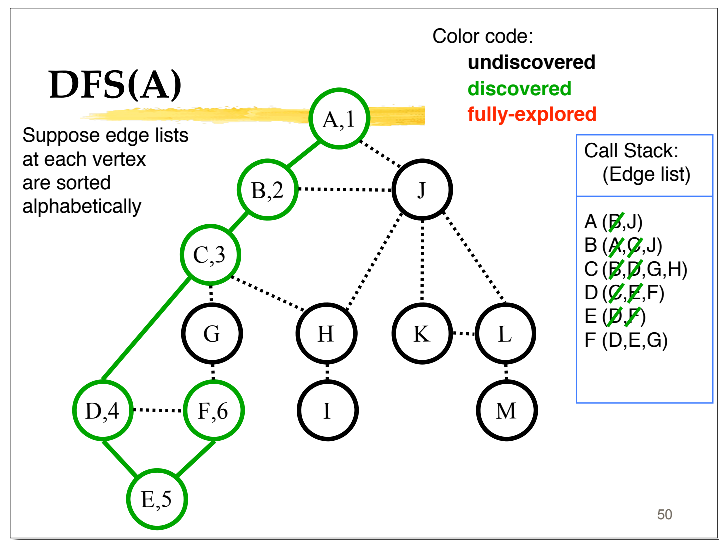
Suppose edge lists at each vertex are sorted alphabetically
A,1
B,2
C,3
D,4
E,5
F
G
H
I
J
K
L
M
Call Stack: (Edge list)
A (B,J)
B (A,C,J)
C (B,D,G,H)
D (C,E,F)
E (D,F)
49


DFS(A)
Color code:
undiscovered
discovered
fully-explored
Suppose edge lists at each vertex are sorted alphabetically
A,1
B,2
C,3
D,4
E,5
F,6
G
H
I
J
K
L
M
Call Stack: (Edge list)
A (B,J)
B (A,C,J)
C (B,D,G,H)
D (C,E,F)
E (D,F)
F (D,E,G)
50

DFS(A)

Color code:
undiscovered
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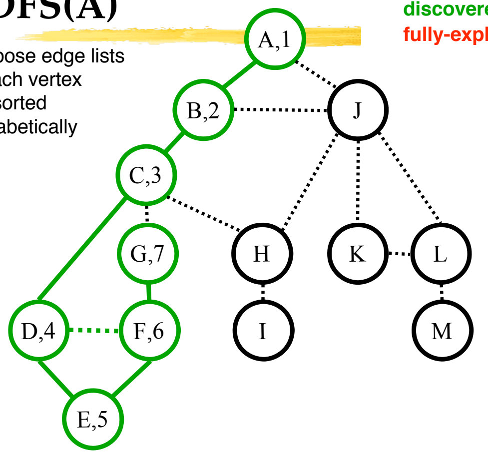
fully-explored

Suppose edge lists at each vertex are sorted alphabetically



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (~~D~~,~~E~~,~~G~~)
G (C,F)

51

DFS(A)

Color code:
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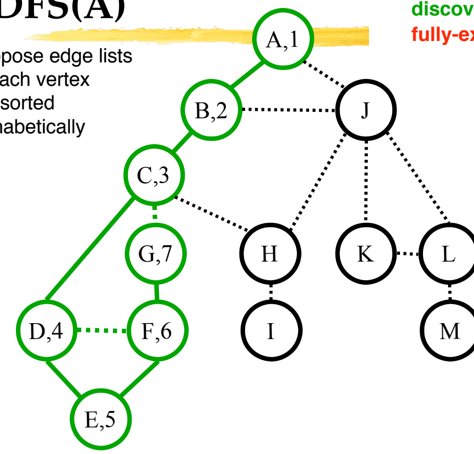

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C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (~~D~~,~~E~~,~~G~~)
G (~~C~~,~~F~~)

52

DFS(A)

Color code:
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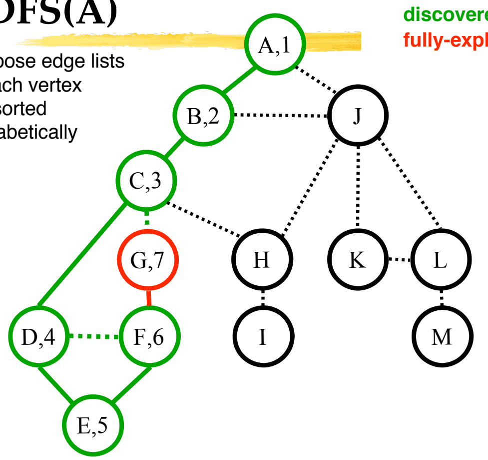

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D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (~~D~~,~~E~~,~~G~~)

53

DFS(A)

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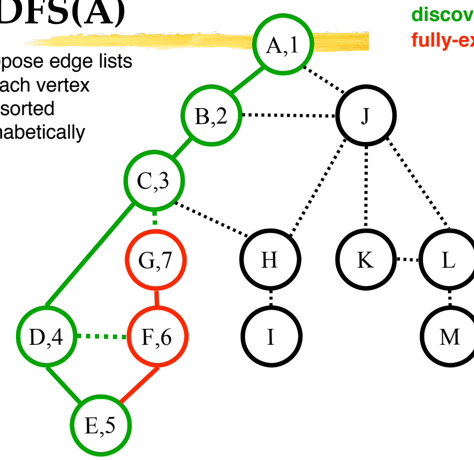

Call Stack:
(Edge list)

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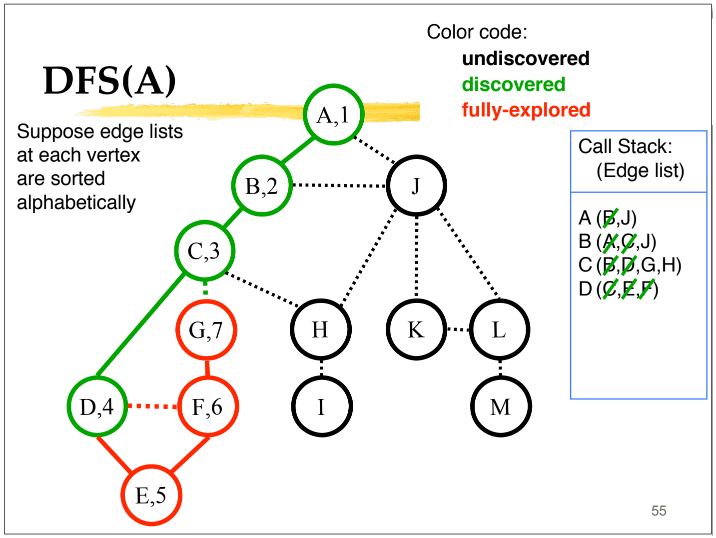
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)

54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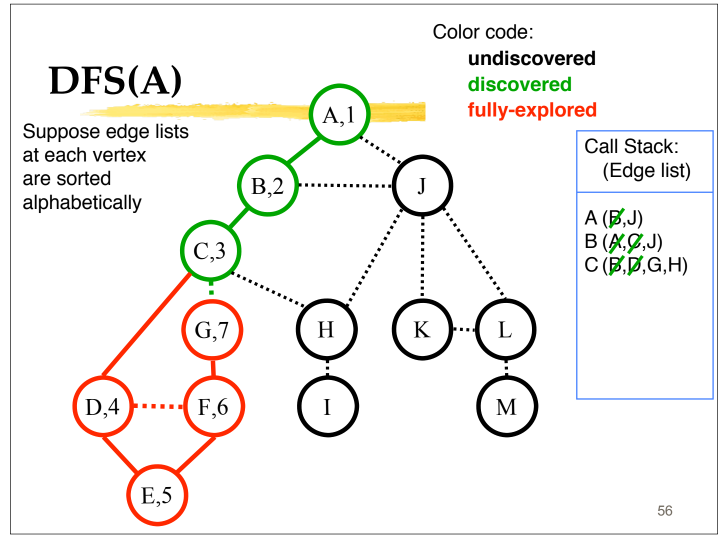
D (C,E,F)
A,1
B,2
C,3
D,4
E,5
F,6
G,7
H
I
J
K
L
M
55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B (A,C,J)
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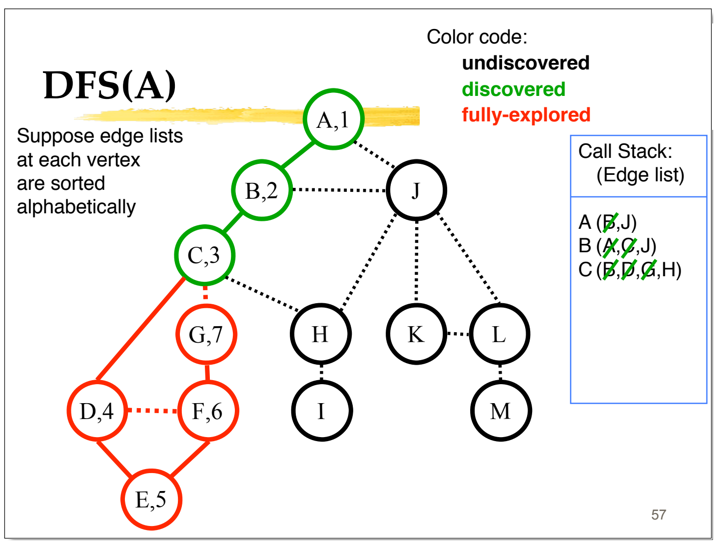
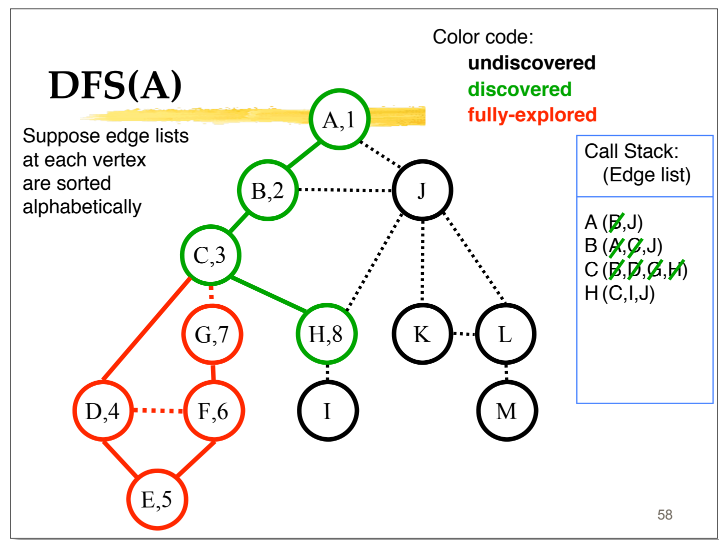
H
I
J
K
L
M
56


DFS(A)
Suppose edge lists at each vertex are sorted alphabetically
Color code:
undiscovered
discovered
fully-explored
Call Stack: (Edge list)
A (B,J)
B (A,C,J)
C (B,D,G,H)
A,1
B,2
C,3
D,4
E,5
F,6
G,7
H
I
J
K
L
M
57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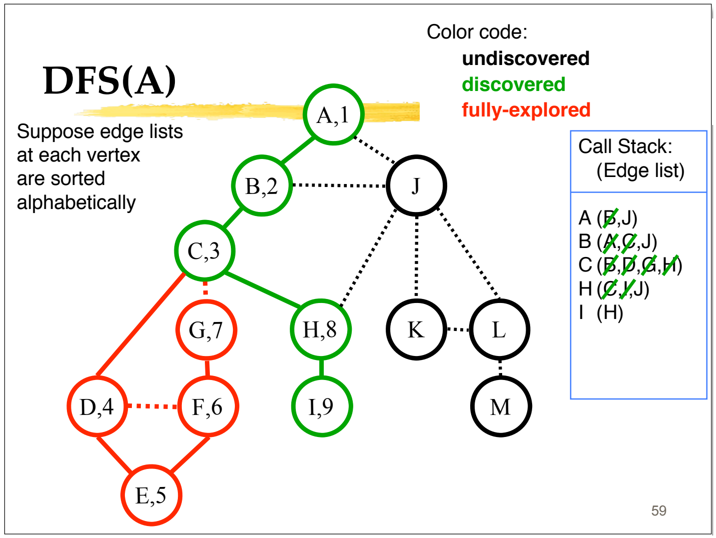
discovered
fully-explored
Call Stack: (Edge list)
A (B,J)
B (A,C,J)
C (B,D,G,H)
H (C,I,J)
A,1
B,2
C,3
D,4
E,5
F,6
G,7
H,8
I
J
K
L
M
58


DFS(A)
Color code:
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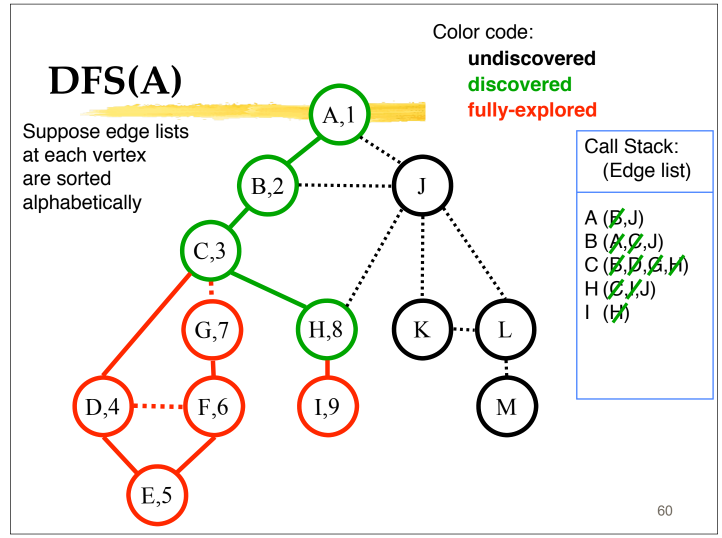
discovered
fully-explored
Suppose edge lists at each vertex are sorted alphabetically
Call Stack: (Edge list)
A (B,J)
B (A,C,J)
C (B,D,G,H)
H (C,I,J)
I (H)
A,1
B,2
J
C,3
G,7
H,8
K
L
D,4
F,6
I,9
M
E,5
59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I (H)
A,1
B,2
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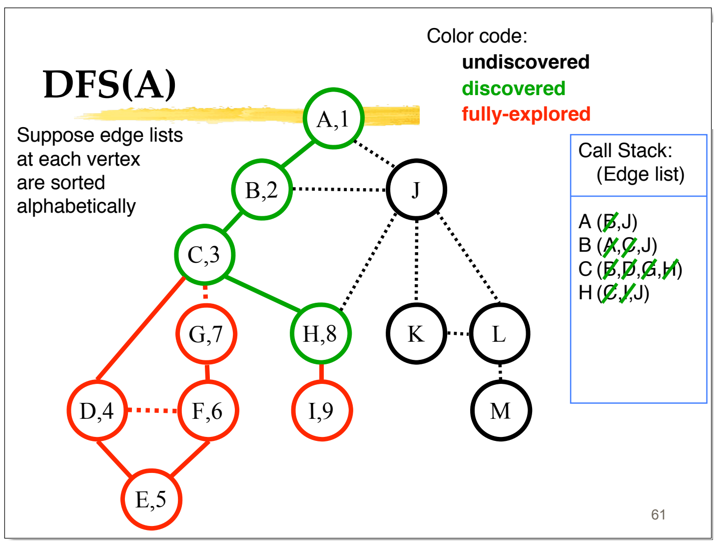
C,3
G,7
H,8
K
L
D,4
F,6
I,9
M
E,5
60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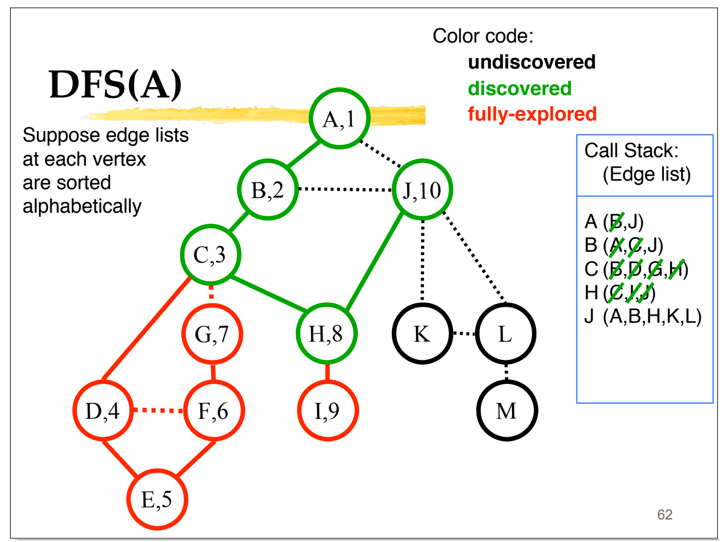
H (C,I,J)
A,1
B,2
J
C,3
G,7
H,8
K
L
D,4
F,6
I,9
M
E,5
61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Color code:
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Suppose edge lists at each vertex are sorted alphabetically
Call Stack: (Edge list)
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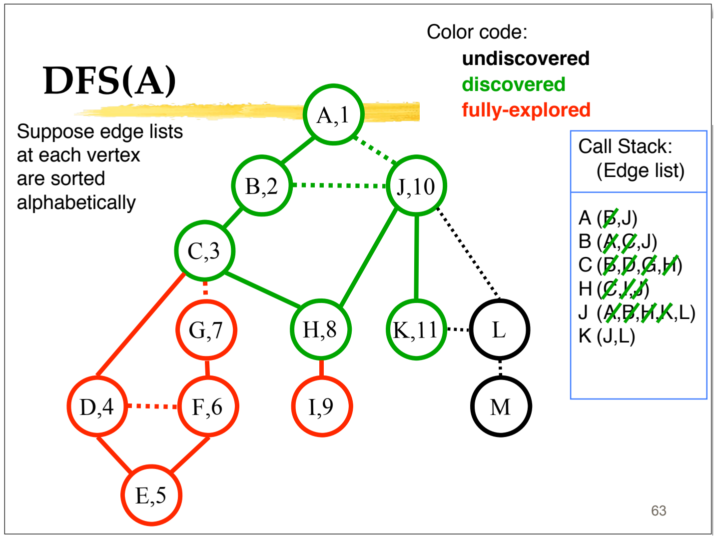
B (A,C,J)
C (B,D,G,H)
H (C,I,J)
J (A,B,H,K,L)
A,1
B,2
J,10
C,3
G,7
H,8
K
L
D,4
F,6
I,9
M
E,5
62


DFS(A)
Color code:
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Call Stack: (Edge list)
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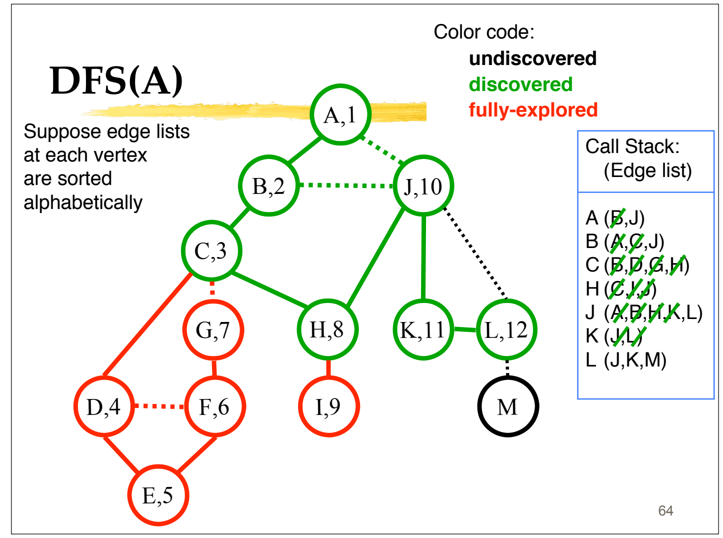
C (B,D,G,H)
H (C,I,J)
J (A,B,H,K,L)
K (J,L)
A,1
B,2
J,10
C,3
G,7
H,8
K,11
L
D,4
F,6
I,9
M
E,5
63


DFS(A)
Color code:
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Call Stack: (Edge list)
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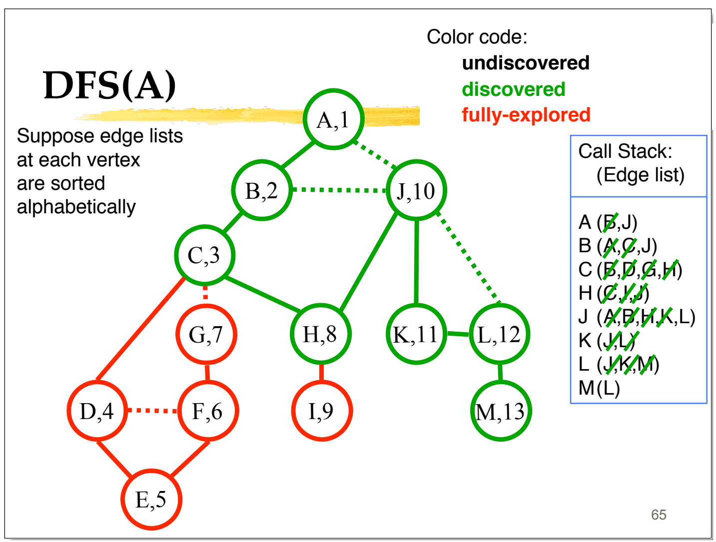
C (B,D,G,H)
H (C,I,J)
J (A,B,H,K,L)
K (J,L)
L (J,K,M)
A,1
B,2
J,10
C,3
G,7
H,8
K,11
L,12
D,4
F,6
I,9
M
E,5
64


DFS(A)
Color code:
undiscovered
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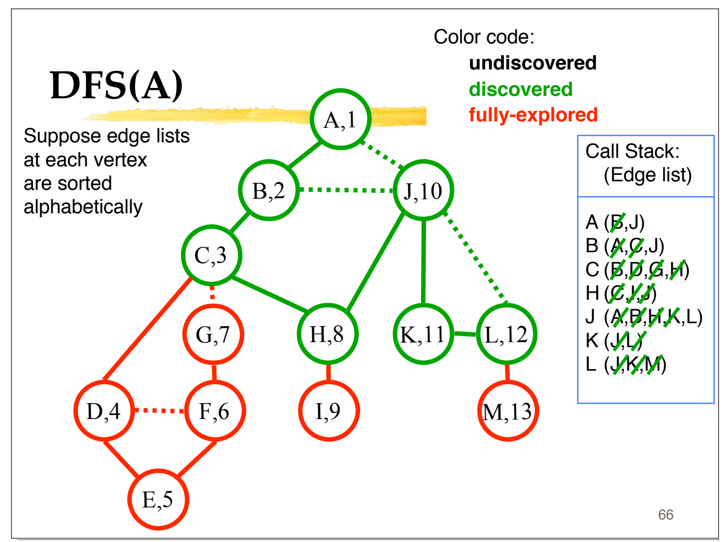
Suppose edge lists at each vertex are sorted alphabetically
Call Stack: (Edge list)
A (B,J)
B (A,C,J)
C (B,D,G,H)
H (C,I,J)
J (A,B,H,K,L)
K (J,L)
L (J,K,M)
M (L)
A,1
B,2
J,10
C,3
G,7
H,8
K,11
L,12
D,4
F,6
I,9
M,13
E,5
65


DFS(A)
Color code:
undiscovered
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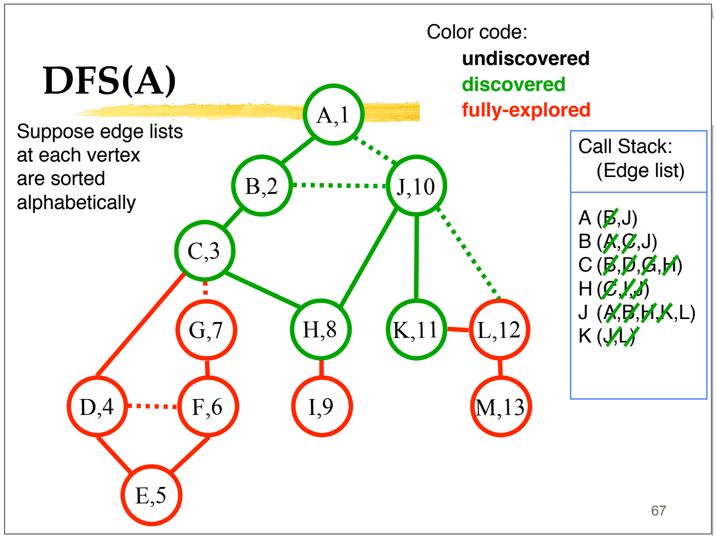
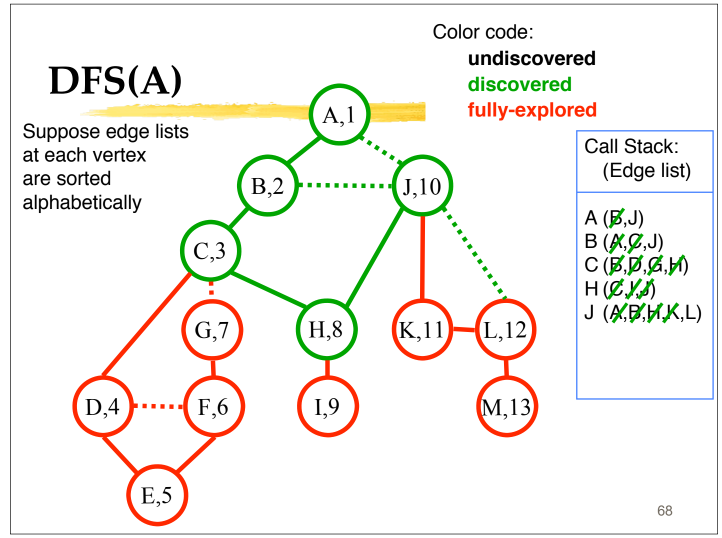
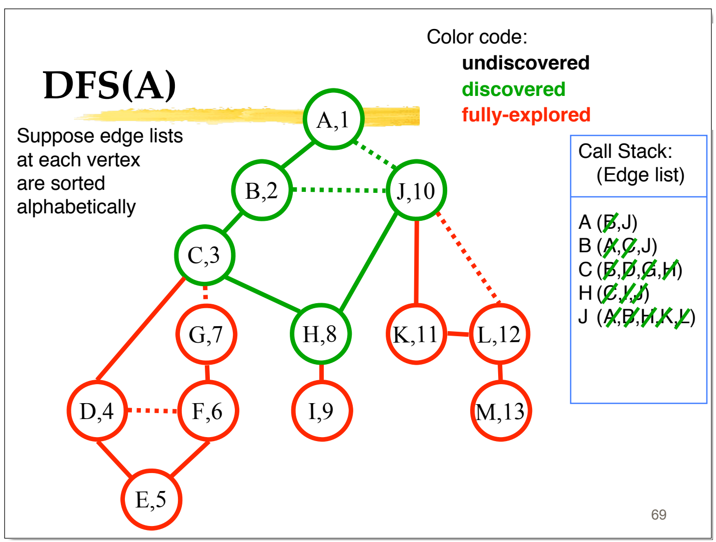
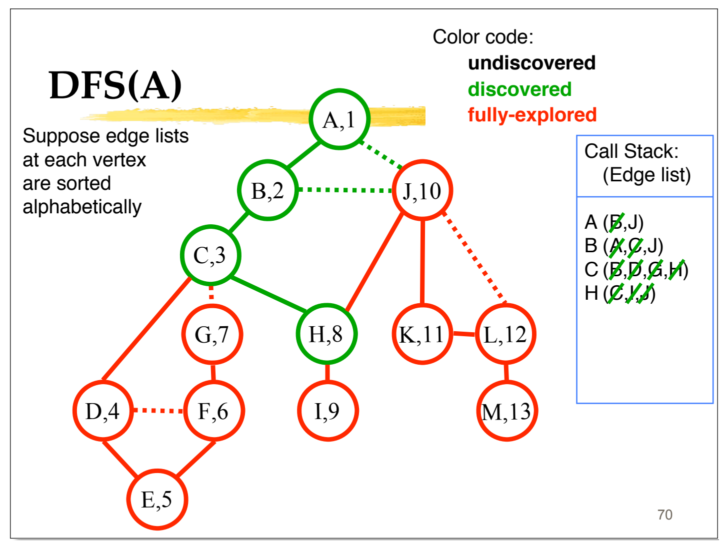
Suppose edge lists at each vertex are sorted alphabetically
Call Stack: (Edge list)
A (B,J)
B (A,C,J)
C (B,D,G,H)
H (C,I,J)
J (A,B,H,K,L)
K (J,L)
L (J,K,M)
A,1
B,2
J,10
C,3
G,7
H,8
K,11
L,12
D,4
F,6
I,9
M,13
E,5
66


DFS(A)
Suppose edge lists at each vertex are sorted alphabetically
Color code:
undiscovered
discovered
fully-explored
A,1
B,2
J,10
C,3
G,7
H,8
K,11
L,12
D,4
F,6
I,9
M,13
E,5
Call Stack:
(Edge list)
A (B,J)
B (A,C,J)
C (B,D,G,H)
H (C,I,J)
J (A,B,H,K,L)
K (J,L)
67

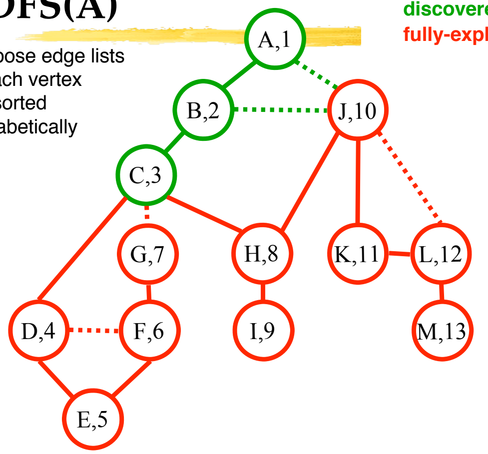

DFS(A)
Suppose edge lists at each vertex are sorted alphabetically
Color code:
undiscovered
discovered
fully-explored
A,1
B,2
J,10
C,3
G,7
H,8
K,11
L,12
D,4
F,6
I,9
M,13
E,5
Call Stack:
(Edge list)
A (B,J)
B (A,C,J)
C (B,D,G,H)
H (C,I,J)
J (A,B,H,K,L)
68


DFS(A)
Suppose edge lists at each vertex are sorted alphabetically
Color code:
undiscovered
discovered
fully-explored
A,1
B,2
J,10
C,3
G,7
H,8
K,11
L,12
D,4
F,6
I,9
M,13
E,5
Call Stack:
(Edge list)
A (B,J)
B (A,C,J)
C (B,D,G,H)
H (C,I,J)
J (A,B,H,K,L)
69


DFS(A)
Suppose edge lists at each vertex are sorted alphabetically
Color code:
undiscovered
discovered
fully-explored
A,1
B,2
J,10
C,3
G,7
H,8
K,11
L,12
D,4
F,6
I,9
M,13
E,5
Call Stack:
(Edge list)
A (B,J)
B (A,C,J)
C (B,D,G,H)
H (C,I,J)
70

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:
undiscovered
discovered
fully-explored

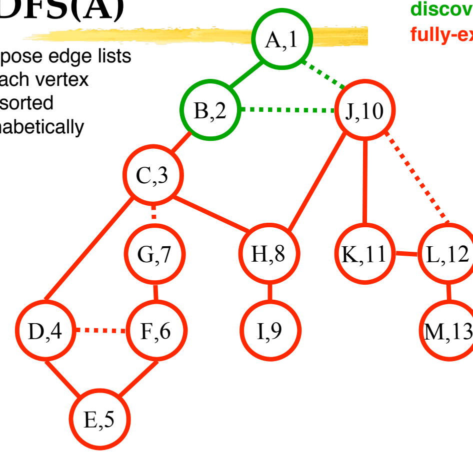
Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)

71

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:
undiscovered
discovered
fully-explored

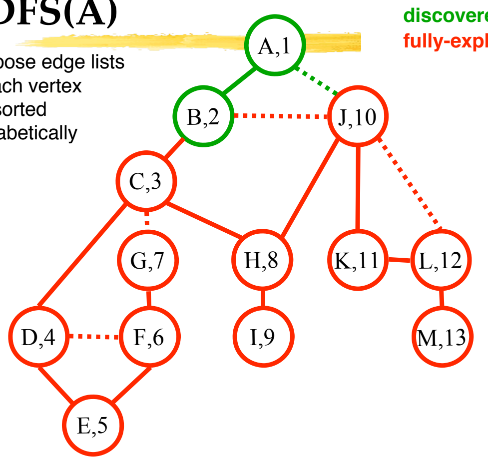
Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)

72

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:
undiscovered
discovered
fully-explored

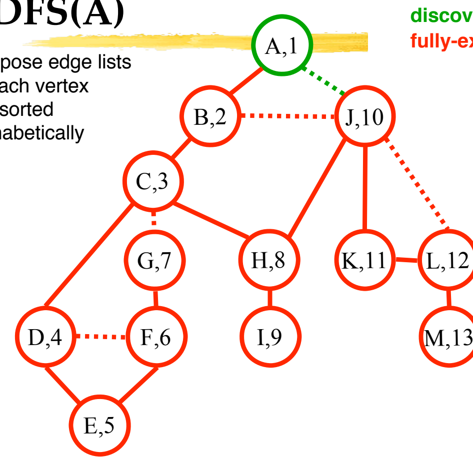
Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,~~J~~)

73

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically

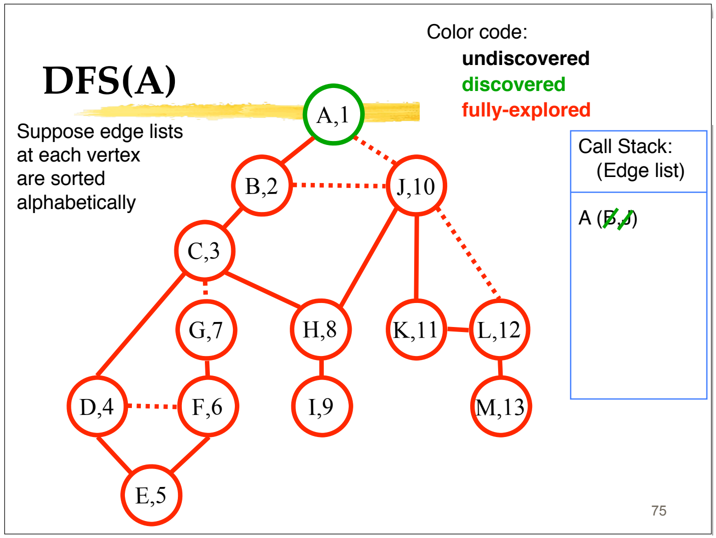


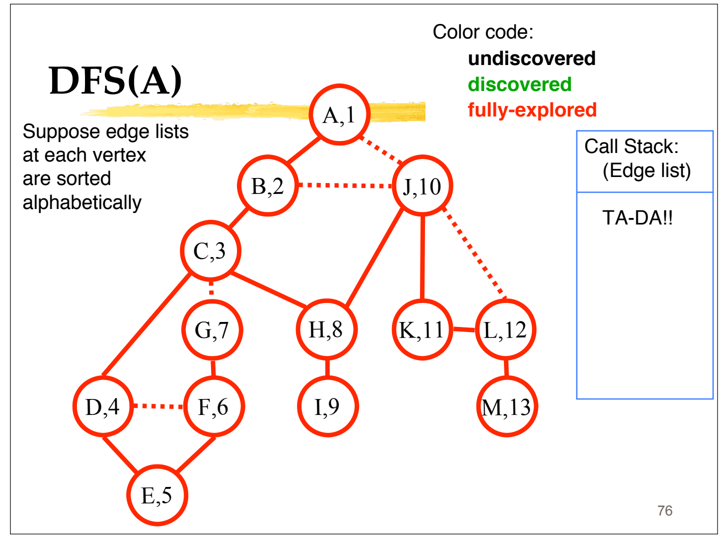
Color code:
undiscovered
discovered
fully-explored

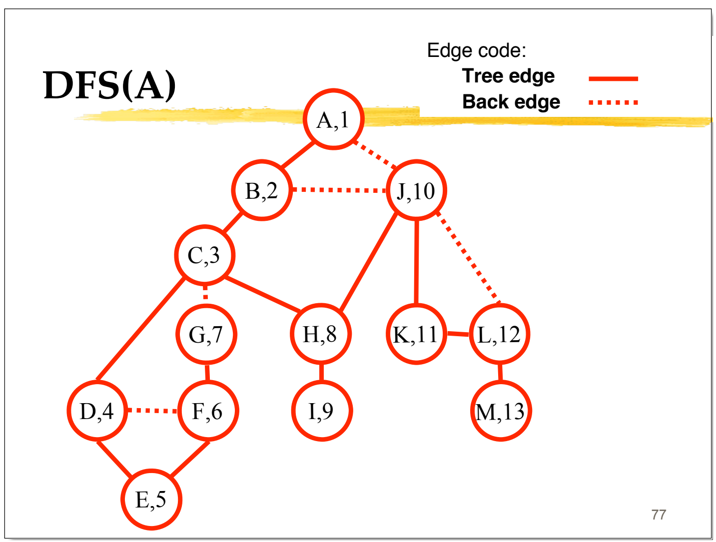
Call Stack:
(Edge list)

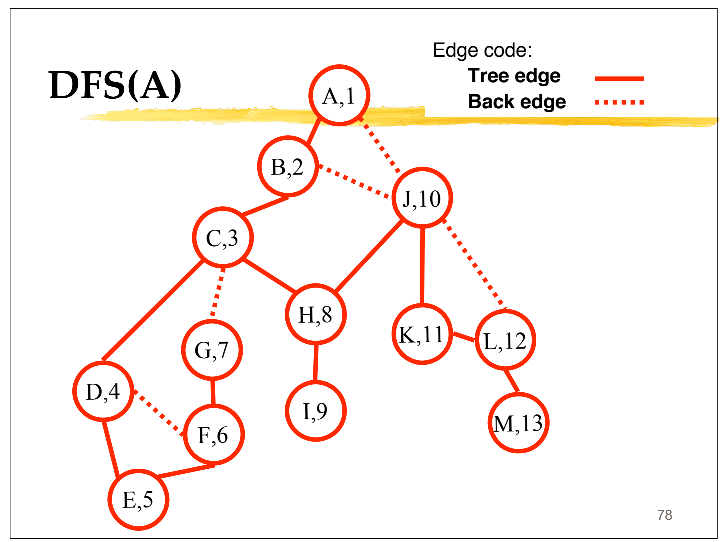
A (~~B~~,J)

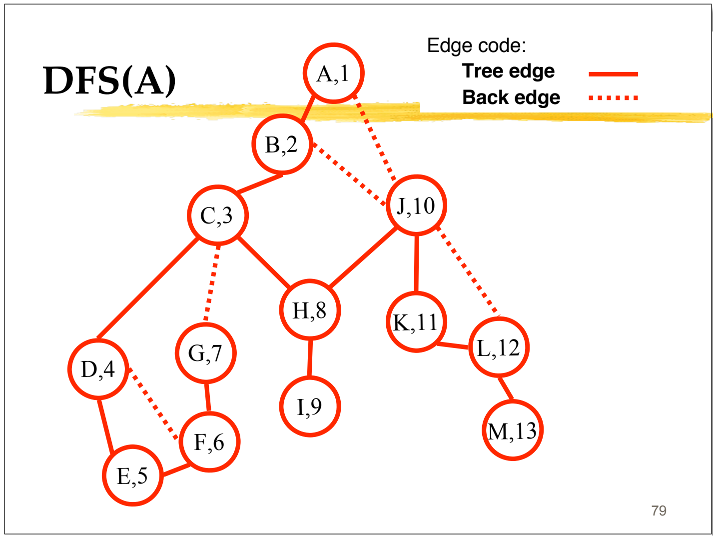
74

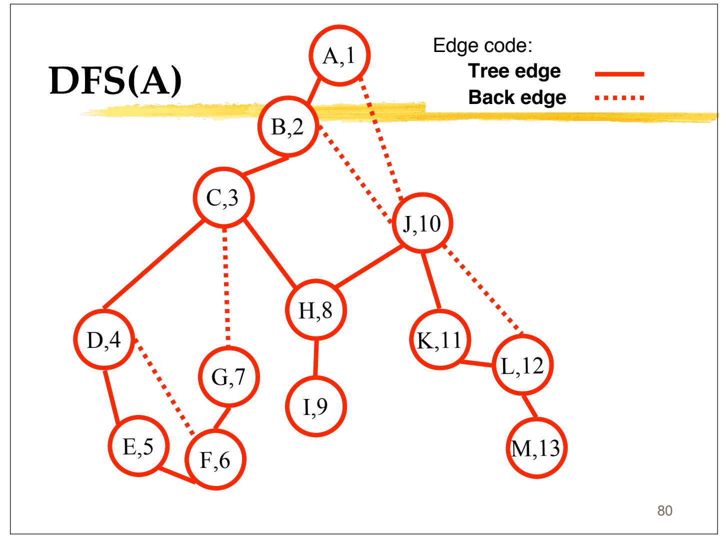

DFS(A)
Color code:
undiscovered
discovered
fully-explored
Suppose edge lists at each vertex are sorted alphabetically
Call Stack: (Edge list)
A (B,J)
A,1
B,2
J,10
C,3
G,7
H,8
K,11
L,12
D,4
F,6
I,9
M,13
E,5
75


DFS(A)
Color code:
undiscovered
discovered
fully-explored
Suppose edge lists at each vertex are sorted alphabetically
Call Stack: (Edge list)
TA-DA!!
A,1
B,2
J,10
C,3
G,7
H,8
K,11
L,12
D,4
F,6
I,9
M,13
E,5
76


DFS(A)
Edge code:
Tree edge
Back edge
A,1
B,2
J,10
C,3
G,7
H,8
K,11
L,12
D,4
F,6
I,9
M,13
E,5
77


DFS(A)
Edge code:
Tree edge
Back edge
A,1
B,2
J,10
C,3
H,8
G,7
K,11
L,12
D,4
F,6
I,9
M,13
E,5
78


DFS(A)
Edge code:
Tree edge
Back edge
A,1
B,2
C,3
D,4
E,5
F,6
G,7
H,8
I,9
J,10
K,11
L,12
M,13
79


DFS(A)
Edge code:
Tree edge
Back edge
A,1
B,2
C,3
D,4
E,5
F,6
G,7
H,8
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J,10
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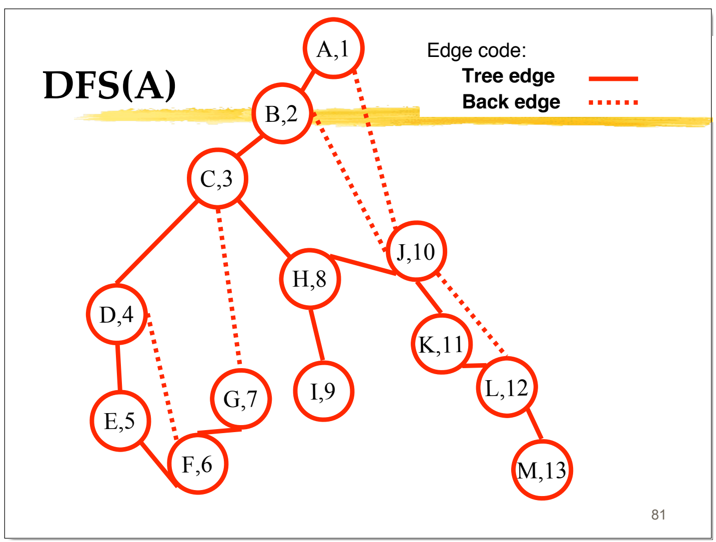
L,12
M,13
80


DFS(A)
Edge code:
Tree edge
Back edge
A,1
B,2
C,3
D,4
E,5
F,6
G,7
H,8
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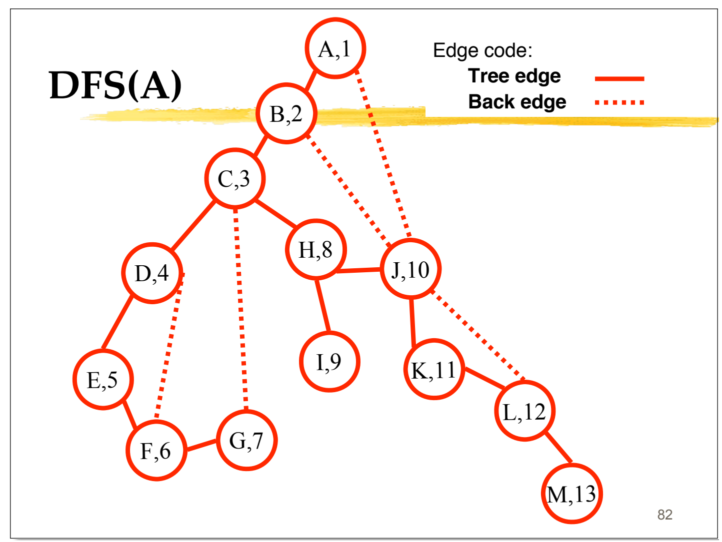
J,10
K,11
L,12
M,13
81


DFS(A)
Edge code:
Tree edge
Back edge
A,1
B,2
C,3
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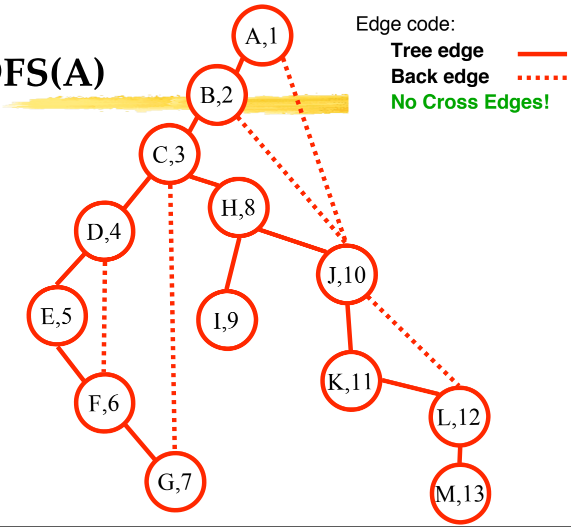
E,5
F,6
G,7
H,8
I,9
J,10
K,11
L,12
M,13
82

DFS(A)

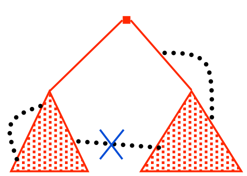


Properties of (Undirected) DFS(v)

- Like BFS(v):
  - DFS(v) visits x if and only if there is a path in G from v to x (through previously unvisited vertices)
  - Edges into then-undiscovered vertices define a ***tree*** – the "depth first spanning tree" of G
- Unlike the BFS tree:
  - the DF spanning tree isn't minimum depth
  - its levels don't reflect min distance from the root
  - non-tree edges never join vertices on the same or adjacent levels
- BUT…

Non-tree edges

- All non-tree edges join a vertex and one of its descendents/ancestors in the DFS tree
- No cross edges!



Why fuss about trees (again)?

- As with BFS, DFS has found a tree in the graph s.t. non-tree edges are “simple”-- only descendant/ancestor

A simple problem on trees

Given: tree T, a value L(v) defined for every vertex v in T
Goal: find M(v), the min value of L(v) anywhere in the subtree rooted at v (including v itself).
How? Depth first search, using:

$$M(v) = \begin{cases} L(v) & \text{if } v \text{ is a leaf} \\ \min(L(v),\ \min_{w \text{ a child of } v} M(w)) & \text{otherwise} \end{cases}$$

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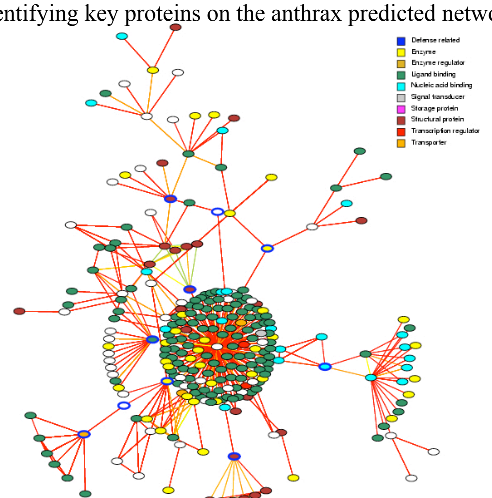
Application: Articulation Points

- A node in an undirected graph is an **articulation point** iff removing it disconnects the graph
- articulation points represent vulnerabilities in a network – single points whose failure would split the network into 2 or more disconnected components

88

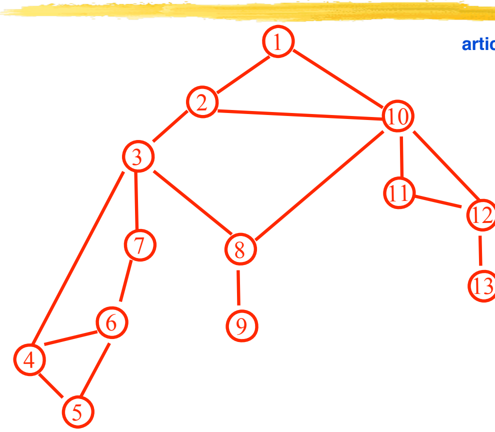
Identifying key proteins on the anthrax predicted network



Articulation point proteins

Ram Samudrala/Jason McDermott

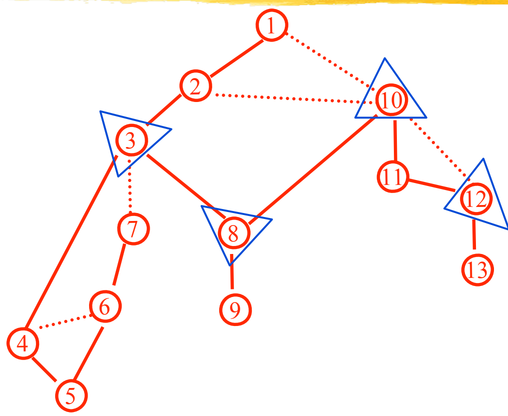
Articulation Points



articulation point iff its removal disconnects the graph

90

Articulation Points



Simple Case: Artic. Pts in a tree

- Leaves -- never articulation points
- Internal nodes -- always articulation points
- Root -- articulation point if and only if two or more children

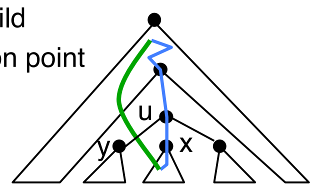
- Non-tree: extra edges remove some articulation points (which ones?)

Articulation Points from DFS

- Root node is an articulation point iff it has more than one child
- Leaf is never an articulation point
- non-leaf, non-root node u is an articulation point

⇕

∃ some child y of u s.t. no non-tree edge goes above u from y or below



If removal of u does NOT separate x, there must be an exit from x's subtree. How? Via back edge.

Articulation Points: the "LOW" function

- Definition: LOW(v) is the lowest dfs# of any vertex that is either in the dfs subtree rooted at v (including v itself) or connected to a vertex in that subtree by a back edge.

trivial

critical

- Key idea 1: if some child x of v has LOW(x) ≥ dfs#(v) then v is an articulation point (excl. root)
- Key idea 2: LOW(v) = min ({dfs#(v)} ∪ {LOW(w) | w a child of v } ∪ { dfs#(x) | {v,x} is a back edge from v })

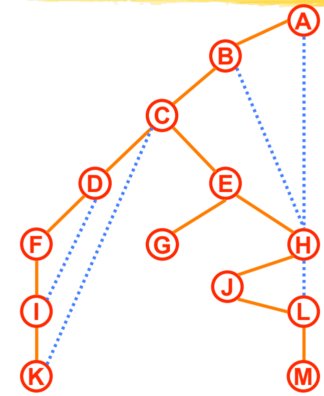
DFS(v) for Finding Articulation Points

```
Global initialization: v.dfs# = -1 for all v.
DFS(v)
 v.dfs# = dfscounter++
 v.low = v.dfs#              // initialization
 for each edge {v,x}
     if (x.dfs# == -1)       // x is undiscovered
        DFS(x)
        v.low = min(v.low, x.low)
        if (x.low >= v.dfs#)
           print "v is art. pt., separating x"
     else if (x is not v's parent)
        v.low = min(v.low, x.dfs#)
```

Except for root. Why?

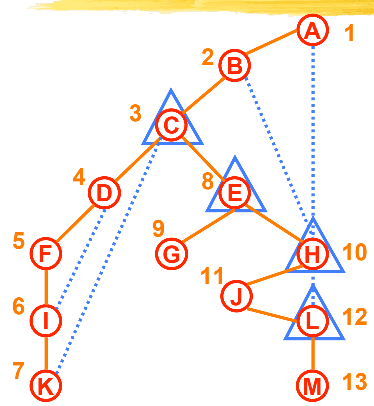
Equiv: "if({v,x} is a back edge)" Why?

Articulation Points



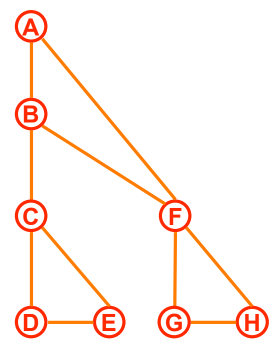
Vertex	DFS #	Low
A		
B		
C		
D		
E		
F		
G		
H		
I		
J		
K		
L		
M		

Articulation Points



Vertex	DFS #	Low
A	1	1
B	2	1
C	3	1
D	4	3
E	8	1
F	5	3
G	9	9
H	10	1
I	6	3
J	11	10
K	7	3
L	12	10
M	13	13

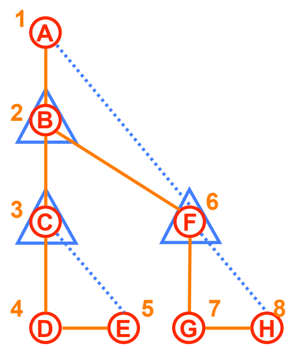
Articulation Points



Vertex	DFS #	Low
A		
B		
C		
D		
E		
F		
G		
H		

AP's:
BCC's:
1)
2)
3)
4)

Articulation Points



Vertex	DFS #	Low
A	1	1
B	2	1
C	3	3
D	4	3
E	5	3
F	6	1
G	7	6
H	8	6

AP's: C, B, F
BCC's:
1) C--D, D--E, E--C
2) B--C
3) A--B, B--F, F--A
4) F--G, G--H, H--F

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