CSE 417: Algorithms and Computational Complexity

Winter 2005

Instructor: W. L. Ruzzo

Lectures 13-17

Divide and Conquer Algorithms

The Divide and Conquer Paradigm

- Outline:
  - General Idea
  - Review of Merge Sort
  - Why does it work?
    - Importance of balance
    - Importance of super-linear growth
  - Two interesting applications
    - Polynomial Multiplication
    - Matrix Multiplication
  - Finding & Solving Recurrences

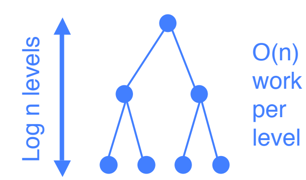
Algorithm Design Techniques

- Divide & Conquer
  - Reduce problem to one or more sub-problems of the same type
  - Typically, each sub-problem is at most a constant fraction of the size of the original problem
    - e.g. Mergesort, Binary Search, Strassen's Algorithm, Quicksort (kind of)

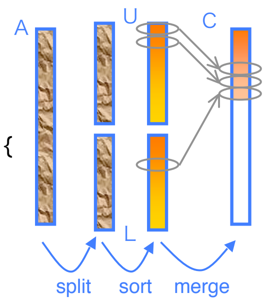
Mergesort (review)

Mergesort: (recursively) sort 2 half-lists, then merge results.

- $T(n)=2T(n/2)+cn, \ n \geq 2$
- $T(1)=0$
- Solution: $\Theta(n \log n)$



Merge Sort




```
MS(A: array[1..n]) returns array[1..n] {
    If(n=1) return A[1];
    New U:array[1:n/2] = MS(A[1..n/2]);
    New L:array[1:n/2] = MS(A[n/2+1..n]);
    Return(Merge(U,L));
    }
Merge(U,L: array[1..n]) {
    New C: array[1..2n];
    a=1; b=1;
    For i = 1 to 2n
      C[i] = “smaller of U[a], L[b] and correspondingly a++ or b++”;
    Return C;
    }
```


Going From Code to Recurrence

1. Carefully define what you’re counting, and write it down!
   “Let C(n) be the number of comparisons between sort keys used by MergeSort when sorting a list of length n ≥ 1”
2. In code, clearly separate **base case** from **recursive case**, highlight **recursive calls,** and **operations being counted.**
3. Write Recurrence(s)

Merge Sort

Base Case

```
MS(A: array[1..n]) returns array[1..n] {
    If(n=1) return A[1];
    New L:array[1:n/2] = MS(A[1..n/2]);
    New R:array[1:n/2] = MS(A[n/2+1..n]);
    Return(Merge(L,R));
    }
Merge(A,B: array[1..n]) {
    New C: array[1..2n];
    a=1; b=1;
    For i = 1 to 2n {
      C[i] = “smaller of A[a], B[b] and a++ or b++”;
    Return C;
    }
```

Recursive calls

Recursive case

Operations being counted

The Recurrence

Base case

$$C(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2C(n/2) + (n-1) & \text{if } n > 1 \end{cases}$$

Recursive calls

One compare per element added to merged list, except the last.

Total time: proportional to C(n)
(loops, copying data, parameter passing, etc.)

Why Balanced Subdivision?

- Alternative "divide & conquer" algorithm:
  - Sort n-1
  - Sort last 1
  - Merge them
- $T(n)=T(n-1)+T(1)+3n$ for $n\geq 2$
- $T(1)=0$
- Solution: $3n + 3(n-1) + 3(n-2) \dots = \Theta(n^2)$

Another D&C Approach

- Suppose we've already invented DumbSort, taking time n^2
- Try *Just One Level* of divide & conquer:
  - DumbSort(first n/2 elements)
  - DumbSort(last n/2 elements)
  - Merge results
- Time: $(n/2)^2 + (n/2)^2 + n = n^2/2 + n$
  - Almost twice as fast!

Another D&C Approach, cont.

- Moral 1:
  Two problems of half size are *better* than one full-size problem, even given the O(n) overhead of recombining, since the base algorithm has *super-linear* complexity.
- Moral 2:
  If a little's good, then more's better—two levels of D&C would be almost 4 times faster, 3 levels almost 8, etc., even though overhead is growing. Best is usually full recursion down to some small constant size (balancing "work" vs "overhead").

Another D&C Approach, cont.

- Moral 3: unbalanced division less good:
  - $(.1n)^2 + (.9n)^2 + n = .82n^2 + n$
    - The 18% savings compounds significantly if you carry recursion to more levels, actually giving O(nlogn), but with a bigger constant. So worth doing if you can't get 50-50 split, but balanced is better if you can.
    - This is intuitively why Quicksort with random splitter is good – badly unbalanced splits are rare, and not instantly fatal.
  - $(1)^2 + (n-1)^2 + n = n^2 - 2n + 2 + n$
    - Little improvement here.

Another D&C Example: Multiplying Faster

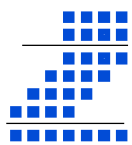
- On the first HW you analyzed our usual algorithm for multiplying numbers
  - $\Theta(n^2)$ time
- We can do better!
  - We'll describe the basic ideas by multiplying polynomials rather than integers
  - Advantage is we don't get confused by worrying about carries at first

Notes on Polynomials

- These are just formal sequences of coefficients so when we show something multiplied by x^k it just means shifted k places to the left – basically no work
- Usual Polynomial Multiplication:

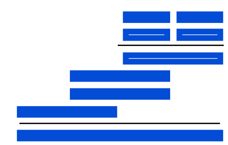
$$\begin{array}{r}
3x^2 + 2x + 2 \\
x^2 - 3x + 1 \\
\hline
3x^2 + 2x + 2 \\
-9x^3 - 6x^2 - 6x \quad\; \\
3x^4 + 2x^3 + 2x^2 \qquad\qquad\quad \\
\hline
3x^4 - 7x^3 - x^2 - 4x + 2
\end{array}$$

Polynomial Multiplication



- Given:
  - Degree $m-1$ polynomials P and Q
    - $P = a_0 + a_1 x + a_2 x^2 + \ldots + a_{m-2}x^{m-2} + a_{m-1}x^{m-1}$
    - $Q = b_0 + b_1 x + b_2 x^2 + \ldots + b_{m-2}x^{m-2} + b_{m-1}x^{m-1}$
- Compute:
  - Degree $2m-2$ Polynomial $P\,Q$
  - $P\,Q = a_0b_0 + (a_0b_1+a_1b_0)\,x + (a_0b_2+a_1b_1+a_2b_0)\,x^2 + \ldots + (a_{m-2}b_{m-1}+a_{m-1}b_{m-2})\,x^{2m-3} + a_{m-1}b_{m-1}\,x^{2m-2}$
- Obvious Algorithm:
  - Compute all a_ib_j and collect terms
  - $\boldsymbol{\Theta(m^2)}$ time

Naive Divide and Conquer

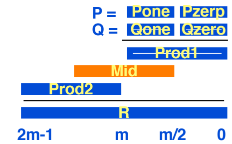


- Assume $m=2k$
  - $P = (a_0 + a_1 x + a_2 x^2 + \ldots + a_{k-2} x^{k-2} + a_{k-1} x^{k-1}) + (a_k + a_{k+1} x + \ldots + a_{m-2}x^{k-2} + a_{m-1}x^{k-1})\,x^k = P_0 + P_1 x^k$
  - $Q = Q_0 + Q_1 x^k$
- $P\,Q = (P_0+P_1x^k)(Q_0+Q_1x^k) = P_0Q_0 + (P_1Q_0+P_0Q_1)x^k + P_1Q_1x^{2k}$
- 4 sub-problems of size $k=m/2$ plus linear combining
  - $T(m)=4T(m/2)+cm$
  - Solution $T(m) = O(m^2)$

Karatsuba's Algorithm

- A better way to compute terms
  - Compute
    - P_0Q_0
    - P_1Q_1
    - $(P_0+P_1)(Q_0+Q_1)$ which is $P_0Q_0+\underline{P_1Q_0+P_0Q_1}+P_1Q_1$
  - Then
    - $P_0Q_1+P_1Q_0 = (P_0+P_1)(Q_0+Q_1) - P_0Q_0 - P_1Q_1$
  - 3 sub-problems of size m/2 plus O(m) work
    - $T(m) = 3\,T(m/2) + cm$
    - $T(m) = O(m^\alpha)$ where $\alpha = \log_2 3 = 1.59...$

Karatsuba: Details




```
PolyMul(P, Q):
    // P, Q are length m =2k vectors, with P[i], Q[i] being
    // the coefficient of x^i in polynomials P, Q respectively.
    if (m==1) return (P[0]*Q[0]);
    Let Pzero be elements 0..k-1 of P; Pone be elements k..m-1
    Qzero, Qone : similar
    Prod1 = PolyMul(Pzero, Qzero);      // result is a (2k-1)-vector
    Prod2 = PolyMul(Pone, Qone);        // ditto
    Pzo = Pzero + Pone;                 // add corresponding elements
    Qzo = Qzero + Qone;                 // ditto
    Prod3 = polyMul(Pzo, Qzo);          // another (2k-1)-vector
    Mid = Prod3 – Prod1 – Prod2;        // subtract corr. elements
    R = Prod1 + Shift(Mid, m/2) + Shift(Prod2,m) // a (2m-1)-vector
    Return( R );
```


Multiplication – The Bottom Line

- Polynomials
  - Naïve: $\Theta(n^2)$
  - Karatsuba: $\Theta(n^{1.59\ldots})$
  - Best known: $\Theta(n \log n)$
    - "Fast Fourier Transform"
- Integers
  - Similar, but some ugly details re: carries, etc. gives $\Theta(n \log n \log\log n)$,
    - but mostly unused in practice

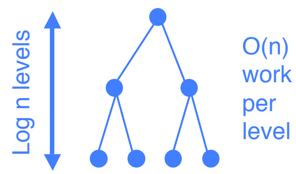
Recurrences

- Where they come from, how to find them (above)
- Next: how to solve them

Mergesort (review)

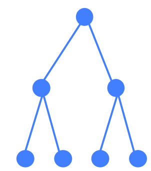
Mergesort: (recursively) sort 2 half-lists, then merge results.

- $T(n)=2T(n/2)+cn$, $n \geq 2$
- $T(1)=0$
- Solution: $\Theta(n \log n)$



26

Solve: $T(1) = c$
$T(n) = 2\,T(n/2) + cn$

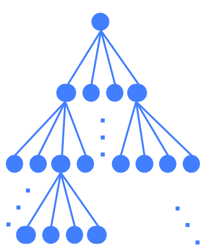


Level	Num	Size	Work
0	$1=2^0$	n	cn
1	$2=2^1$	$n/2$	$2\ c\ n/2$
2	$4=2^2$	$n/4$	$4\ c\ n/4$
…	…	…	…
i	2^i	$n/2^i$	$2^i\ c\ n/2^i$
…	…	…	…
$k-1$	2^{k-1}	$n/2^{k-1}$	$2^{k-1}\ c\ n/2^{k-1}$
k	2^k	$n/2^k=1$	$2^k\ T(1)$

Total work: add last col

27

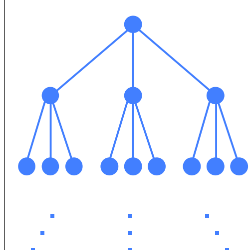
Solve: $T(1) = c$
$T(n) = 4\,T(n/2) + cn$



Level	Num	Size	Work
0	$1=4^0$	n	cn
1	$4=4^1$	$n/2$	$4\ c\ n/2$
2	$16=4^2$	$n/4$	$16\ c\ n/4$
…	…	…	…
i	4^i	$n/2^i$	$4^i\ c\ n/2^i$
…	…	…	…
$k-1$	4^{k-1}	$n/2^{k-1}$	$4^{k-1}\ c\ n/2^{k-1}$
k	4^k	$n/2^k=1$	$4^k\ T(1)$

28

Solve: $T(1) = c$
$T(n) = 3\,T(n/2) + cn$



Level	Num	Size	Work
0	$1=3^0$	n	cn
1	$3=3^1$	$n/2$	$3\ c\ n/2$
2	$9=3^2$	$n/4$	$9\ c\ n/4$
…	…	…	…
i	3^i	$n/2^i$	$3^i\ c\ n/2^i$
…	…	…	…
$k-1$	3^{k-1}	$n/2^{k-1}$	$3^{k-1}\ c\ n/2^{k-1}$
k	3^k	$n/2^k=1$	$3^k\ T(1)$

$n = 2^k$; $k = \log_2 n$

Total Work: $T(n) = \sum_{i=0}^{k} 3^i cn / 2^i$

29

Solve: T(1) = c
T(n) = 3 T(n/2) + cn (cont.)

$$T(n) = \sum_{i=0}^{k} 3^i cn/2^i$$
$$= cn \sum_{i=0}^{k} 3^i/2^i$$
$$= cn \sum_{i=0}^{k} \left(\tfrac{3}{2}\right)^i$$
$$= cn \frac{\left(\tfrac{3}{2}\right)^{k+1} - 1}{\left(\tfrac{3}{2}\right) - 1}$$

$$\sum_{i=0}^{k} x^i = \frac{x^{k+1} - 1}{x - 1} \quad (x \neq 1)$$

30

Solve: T(1) = c
T(n) = 3 T(n/2) + cn (cont.)

$$= 2cn\left(\left(\tfrac{3}{2}\right)^{k+1} - 1\right)$$
$$< 2cn\left(\tfrac{3}{2}\right)^{k+1}$$
$$= 3cn\left(\tfrac{3}{2}\right)^{k}$$
$$= 3cn\frac{3^k}{2^k}$$

31

Solve: T(1) = c
T(n) = 3 T(n/2) + cn (cont.)

$$= 3cn\frac{3^{\log_2 n}}{2^{\log_2 n}}$$
$$= 3cn\frac{3^{\log_2 n}}{n}$$
$$= 3c\,3^{\log_2 n}$$
$$= 3c\left(n^{\log_2 3}\right)$$
$$\boxed{= O\left(n^{1.59\ldots}\right)}$$

$$a^{\log_b n} = \left(b^{\log_b a}\right)^{\log_b n} = \left(b^{\log_b n}\right)^{\log_b a} = n^{\log_b a}$$

32

Master Divide and Conquer Recurrence

- If $T(n) = aT(n/b)+cn^k$ for $n > b$ then
  - if $a > b^k$ then $T(n)$ is $\Theta(n^{\log_b a})$
  - if $a < b^k$ then $T(n)$ is $\Theta(n^k)$
  - if $a = b^k$ then $T(n)$ is $\Theta(n^k \log n)$
- Works even if it is $\lceil n/b \rceil$ instead of n/b.

33

Another Example:

Matrix Multiplication –

Strassen's Method

Multiplying Matrices

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \bullet \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}
$$

$$
= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{14}b_{41} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} + a_{14}b_{42} & \circ & a_{11}b_{14} + a_{12}b_{24} + a_{13}b_{34} + a_{14}b_{44} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} + a_{24}b_{41} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} + a_{24}b_{42} & \circ & a_{21}b_{14} + a_{22}b_{24} + a_{23}b_{34} + a_{24}b_{44} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} + a_{34}b_{41} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} + a_{34}b_{42} & \circ & a_{31}b_{14} + a_{32}b_{24} + a_{33}b_{34} + a_{34}b_{44} \\ a_{41}b_{11} + a_{42}b_{21} + a_{43}b_{31} + a_{44}b_{41} & a_{41}b_{12} + a_{42}b_{22} + a_{43}b_{32} + a_{44}b_{42} & \circ & a_{41}b_{14} + a_{42}b_{24} + a_{43}b_{34} + a_{44}b_{44} \end{bmatrix}
$$

- n^3 multiplications, n^3-n^2 additions

Simple Matrix Multiply

```
for i = 1 to n
   for j = I to n
      C[i,j] = 0
      for k = 1 to n
         C[i,j] = C[i,j] + A[i,k] * B[k,j]
```

n^3 multiplications, n^3-n^2 additions

Multiplying Matrices

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \bullet \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}
$$

$$
= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{14}b_{41} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} + a_{14}b_{42} & \circ & a_{11}b_{14} + a_{12}b_{24} + a_{13}b_{34} + a_{14}b_{44} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} + a_{24}b_{41} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} + a_{24}b_{42} & \circ & a_{21}b_{14} + a_{22}b_{24} + a_{23}b_{34} + a_{24}b_{44} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} + a_{34}b_{41} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} + a_{34}b_{42} & \circ & a_{31}b_{14} + a_{32}b_{24} + a_{33}b_{34} + a_{34}b_{44} \\ a_{41}b_{11} + a_{42}b_{21} + a_{43}b_{31} + a_{44}b_{41} & a_{41}b_{12} + a_{42}b_{22} + a_{43}b_{32} + a_{44}b_{42} & \circ & a_{41}b_{14} + a_{42}b_{24} + a_{43}b_{34} + a_{44}b_{44} \end{bmatrix}
$$

Multiplying Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \bullet \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}+a_{14}b_{42} & \circ & a_{11}b_{14}+a_{12}b_{24}+a_{13}b_{34}+a_{14}b_{44} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}+a_{24}b_{41} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}+a_{24}b_{42} & \circ & a_{21}b_{14}+a_{22}b_{24}+a_{23}b_{34}+a_{24}b_{44} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}+a_{34}b_{41} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}+a_{34}b_{42} & \circ & a_{31}b_{14}+a_{32}b_{24}+a_{33}b_{34}+a_{34}b_{44} \\ a_{41}b_{11}+a_{42}b_{21}+a_{43}b_{31}+a_{44}b_{41} & a_{41}b_{12}+a_{42}b_{22}+a_{43}b_{32}+a_{44}b_{42} & \circ & a_{41}b_{14}+a_{42}b_{24}+a_{43}b_{34}+a_{44}b_{44} \end{bmatrix}$$

38

Multiplying Matrices

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \bullet \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{bmatrix}$$

39

Multiplying Matrices

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{bmatrix}$$

- Counting arithmetic operations:
  $T(n) = 8T(n/2) + 4(n/2)^2 = 8T(n/2) + n^2$

40

Multiplying Matrices

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 8T(n/2) + n^2 & \text{if } n > 1 \end{cases}$$

- By Master Recurrence, if
  $T(n) = aT(n/b)+cn^k$ & $a > b^k$ then
  $T(n) = \Theta(n^{\log_b a}) = \Theta(n^{\log_2 8}) = \Theta(n^3)$

41

Strassen's algorithm

- Strassen's algorithm
  - Multiply 2x2 matrices using 7 instead of 8 multiplications (and lots more than 4 additions)
  - $T(n)=7\,T(n/2)+cn^2$
    - $7>2^2$ so $T(n)$ is $\Theta(n^{\log_2 7})$ which is $O(n^{2.81})$
  - Fastest algorithms theoretically use $O(n^{2.376})$ time
    - not practical but Strassen's is practical provided calculations are exact and we stop recursion when matrix has size about 100 (maybe 10)

42

The algorithm

$P_1 = A_{12}(B_{11}+B_{21})$ $\quad$ $P_2 = A_{21}(B_{12}+B_{22})$

$P_3 = (A_{11} - A_{12})B_{11}$ $\quad$ $P_4 = (A_{22} - A_{21})B_{22}$

$P_5 = (A_{22} - A_{12})(B_{21} - B_{22})$

$P_6 = (A_{11} - A_{21})(B_{12} - B_{11})$

$P_7 = (A_{21} - A_{12})(B_{11}+B_{22})$

$C_{11}= P_1+P_3$ $\quad$ $C_{12} = P_2+P_3+P_6-P_7$

$C_{21}= P_1+P_4+P_5+P_7$ $\quad$ $C_{22} = P_2+P_4$

43

Another D&C Example: Fast exponentiation

- Power(a,n)
  - **Input:** integer **n** and number **a**
  - **Output:** $\mathbf{a^n}$
- Obvious algorithm
  - **n-1** multiplications
- Observation:
  - if **n** is even, **n=2m**, then $\mathbf{a^n=a^m \cdot a^m}$

44

Divide & Conquer Algorithm

- Power(a,n)

```
    if n=0 then
        return(1)
    else
        x ←Power(a,⌊n/2⌋)
        if n is even then
            return(x•x)
        else
            return(a•x•x)
```

45

Analysis

- Worst-case recurrence
  - $T(n) = T(\lfloor n/2 \rfloor)+2$
- By master theorem
  - $T(n) = O(\log n)$ (a=1, b=2, k=0)
- More precise analysis:
  - $T(n)= \lceil \log_2 n \rceil$ + # of 1's in n's binary representation

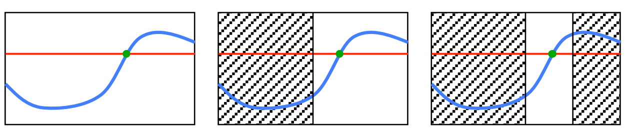
46

A Practical Application- RSA

- Instead of $\mathbf{a^n}$ want $\mathbf{a^n}$ **mod N**
  - $\mathbf{a^{i+j}}$ mod **N** = (($\mathbf{a^i}$ mod **N**)•($\mathbf{a^j}$ mod **N**)) mod **N**
  - same algorithm applies with each **x•y** replaced by
    - ((**x** mod **N**)•(**y** mod **N**)) mod **N**
- In RSA cryptosystem (widely used for security)
  - need $\mathbf{a^n}$ **mod N** where **a**, **n**, **N** each typically have 1024 bits
  - Power: at most 2048 multiplies of 1024 bit numbers
    - relatively easy for modern machines
  - Naive algorithm: 2^{1024} multiplies

47

Another Example: Binary search for roots (bisection method)



- Given:
  - continuous function **f** and two points **a<b** with **f(a)<0** and **f(b)>0**
- Find:
  - approximation to **c** s.t. **f(c)=0** and **a<c<b**

48

Divide and Conquer Summary

- Powerful technique, when applicable
- Divide large problem into a few smaller problems of the same type
- Choosing subproblems of roughly equal size is usually critical
- Examples:
  - Merge sort, quicksort (sort of), polynomial multiplication, FFT, Strassen's matrix multiplication algorithm, powering, binary search, root finding by bisection, …

49