CSE 417: Algorithms and Computational Complexity

5: Dynamic Programming, II
Linear Partition

Winter 2005

W. L. Ruzzo

Dynamic Programming

- Useful when
  - Same recursive sub-problems occur repeatedly
  - Can anticipate them
  - Can find solution to whole problem without knowing internal details of sub-problem solutions
    - "principle of optimality"

List partition problem

- Given: a sequence of n positive integers $s_1,\ldots,s_n$ and a positive integer k
- Find: a partition of the list into up to k blocks:

  $s_1,\ldots,s_{i_1}|s_{i_1+1}\ldots s_{i_2}|s_{i_2+1}\ldots\ s_{i_{k-1}}\ |s_{i_{k-1}+1}\ldots s_n$

  so that the *sum of the numbers in the largest block is as small as possible*.

  i.e., find spots for up to $k-1$ dividers

Greedy approach

- Ideal size would be P = $\sum_{i=1}^{n} s_i/k$
- Greedy: walk along until what you have so far adds up to ≥ P then insert a divider
- Problem: it may not exact (or correct)

  100 200 400 500 900 700 600 800 600 , k=3

  - sum is 4800 so size must be at least 1600.
  - Greedy? Best?

Recursive solution

- Try all possible values for the position of the *last* divider
- For each position of this last divider
  - – there are k-2 other dividers that must divide the list of numbers prior to the last divider as evenly as possible
    - $s_1,\ldots,s_{i_1}|s_{i_1+1}\ldots s_{i_2}|s_{i_2+1}\ldots s_{i_{k-1}}|s_{i_{k-1}+1}\ldots s_n$
  - – recursive sub-problem of the same type

Recursive idea

- Let $M[n,k]$ the smallest cost (size of largest block) of any partition of the n into k pieces.
- If best position for last divider lies between the i^{th} and $i+1^{st}$ then

$$M[n,k] = \max\left(\overset{\text{max cost of 1st k-1 blocks}}{M[i,k-1]} \; , \; \overset{\text{cost of last block}}{\sum_{j=i+1}^{n} s_j} \right)$$

- In general

$$M[n,k] = \min_{i<n} \max\left(M[i,k-1] \; , \; \sum_{j=i+1}^{n} s_j \right)$$

- Base case(s)?

Time-saving - prefix sums

- Computing the costs of the blocks may be expensive and involved repeated work
- Idea: Pre-compute prefix sums
- Length of block

  $s_{i+1}+\ldots+s_j$

  is just

  p[j]-p[i]

- Cost: n additions

$p[1]=s_1$
$p[2]=s_1+s_2$
$p[3]=s_1+s_2+s_3$

...
$p[n]=s_1+s_2+\ldots+s_n$

Linear Partition Algorithm

```
Partition(S,k):
    p[0]←0; for i=1 to n do p[i] ←p[i-1]+$s_i$
    for i=1 to n do M[i,1] ←p[i]
    for j=1 to k do M[1,j] ← $s_1$
    for i=2 to n do
        for j=2 to k do
            M[i,j] ← $\min_{pos<i}\{\max(M[pos,j-1], \overbrace{p[i]-p[pos]}^{\sum_{j=pos+1}^{i} s_j})\}$
```

Trace-Back: *Finding* Solns

- Above gives *value* of best solution
- Q: How do you *find* it?

- A: work backwards from answer

Linear Partition Algorithm

```
Partition(S,k):
    p[0]←0; for i=1 to n do p[i] ←p[i-1]+$s_i$
    for i=1 to n do M[i,1] ←p[i]
    for j=1 to k do M[1,j] ← $s_1$
    for i=2 to n do
        for j=2 to k do
            M[i,j] ← $\min_{pos<i}$ {max(M[pos,j-1], p[i]-p[pos])}
            D[i,j] ← value of pos where min is achieved
```

Linear Partition Algorithm

```
Partition(S,k):
p[0]←0; for i=1 to n do p[i] ←p[i-1]+$s_i$
    for i=1 to n do M[i,1] ←p[i]
    for j=1 to k do M[1,j] ← $s_1$
    for i=2 to n do
        for j=2 to k do
            M[i,j]←∞
            for pos=1 to i-1 do
                s←max(M[pos,j-1], p[i]-p[pos])
                if M[i,j]>s then
                    M[i,j] ←s ; D[i,j] ←pos
```

Example:

	1	2	3
100			
200			
400			
500			
900			
700			
600			
800			
600			

Example:

	1	**2**	**3**
100	100	100	100
200	300		
400	700		
500	1200		
900	2100		
700	2800		
600	3400		
800	4200		
600	4800		

Example:

	1	**2**	**3**
100	100	100	100
200	300	200	200
400	700	400	400
500	1200	700	500
900	2100	1200	900
700	2800	1600	1400
600	3400	2100	
800	4200	2100	
600	4800	2700	

Exercises

- Finish example
- Make up another example & try it
- Figure out from example(s) where the dividers go
- Write an algorithm that, based on the M & D matrices, figures out where the dividers go

Goals: Skills to learn

- Recognize when dynamic programming is a plausible approach
  - E.g., recursive formulation, repeated subproblems, Global opt depends on opt subsolution, but not details thereof.
- Understand the logic of the correctness of the method from the recurrence & vice versa
- Construct D.P. algorithms for new problems you see