CSE 417: Algorithms and Computational Complexity

3: Complexity

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Efficiency

- Our correct TSP algorithm was incredibly slow
- Basically slow no matter what computer you have
- We would like a general theory of "efficiency" that is
  - Simple
  - Relatively independent of changing technology
  - But still useful for prediction - "theoretically bad" algorithms should be bad in practice and vice versa (usually)

Measuring efficiency: The RAM model

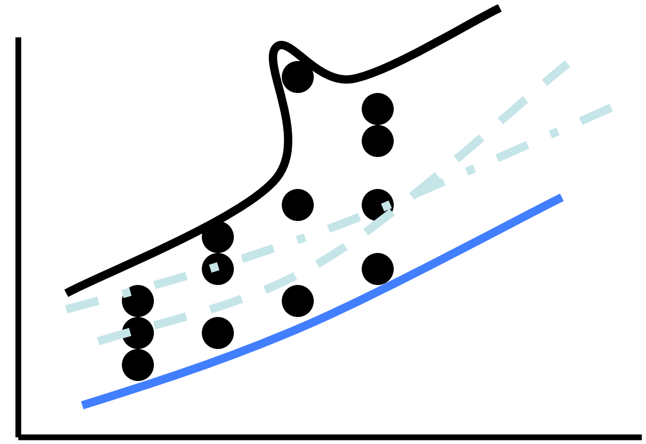
- RAM = Random Access Machine

- Time ≈ # of instructions executed in an ideal assembly language
  - each simple operation (+,*,-,=,if,call) takes one time step
  - each memory access takes one time step
- No bound on the memory

We left out things but...

- Things we’ve dropped
  - memory hierarchy
    - disk, caches, registers have many orders of magnitude differences in access time
  - not all instructions take the same time in practice
- However,
  - the RAM model is useful for designing algorithms and measuring their efficiency
  - one can usually tune implementations so that the hierarchy etc. is not a huge factor

Complexity analysis



- Problem size n
  - **Worst-case complexity**: **max** # steps algorithm takes on any input of size n
  - **Best-case complexity: min** # steps algorithm takes on any input of size n
  - **Average-case complexity**: **avg** # steps algorithm takes on inputs of size n

Pros and cons:

- Best-case
  - unrealistic overselling
  - can "cheat": tune algorithm for one easy input
- Worst-case
  - a fast algorithm has a comforting guarantee
  - no way to cheat by hard-coding special cases
  - maybe too pessimistic
- Average-case
  - over what probability distribution?
  - different people may have different average problems

Why Worst-Case Analysis?

- Appropriate for time-critical applications, e.g. avionics
- Unlike Average-Case, no debate about what the right definition is
- Analysis often easier
- Result is often representative of "typical" problem instances
- Of course there are exceptions…

General Goals

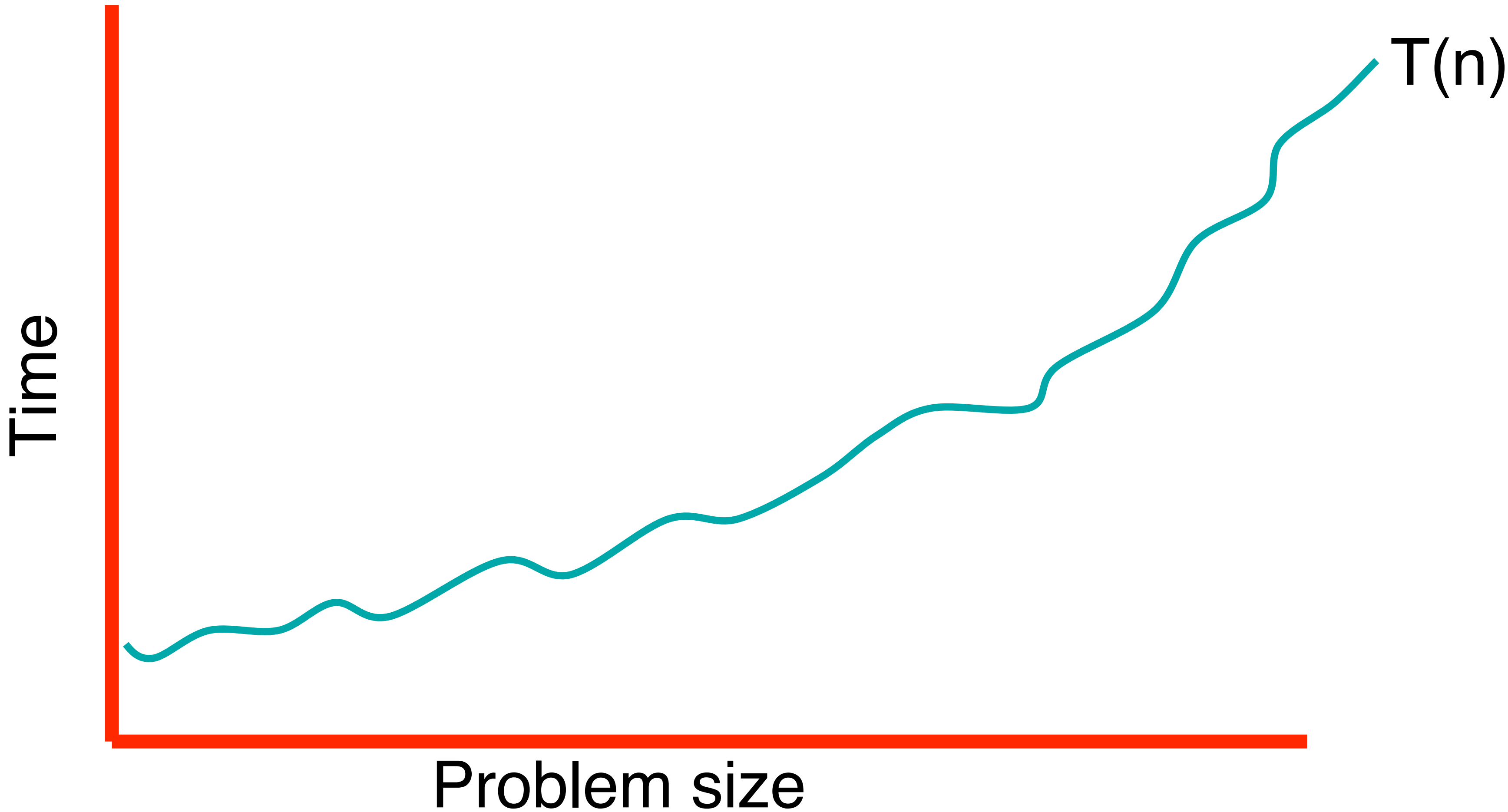
- Characterize *growth rate* of run time as a function of problem size, up to a *constant factor*
- Why not try to be more precise?
  - Technological variations (computer, compiler, OS, …) easily 10x or more
  - Being more precise is a ton of work
  - A key question is "scale up": if I can afford to do it today, how much longer will it take when my business problems are twice as large? (E.g. today: cn^2, next year: $c(2n)^2 = 4cn^2$: 4 x longer.)

Complexity

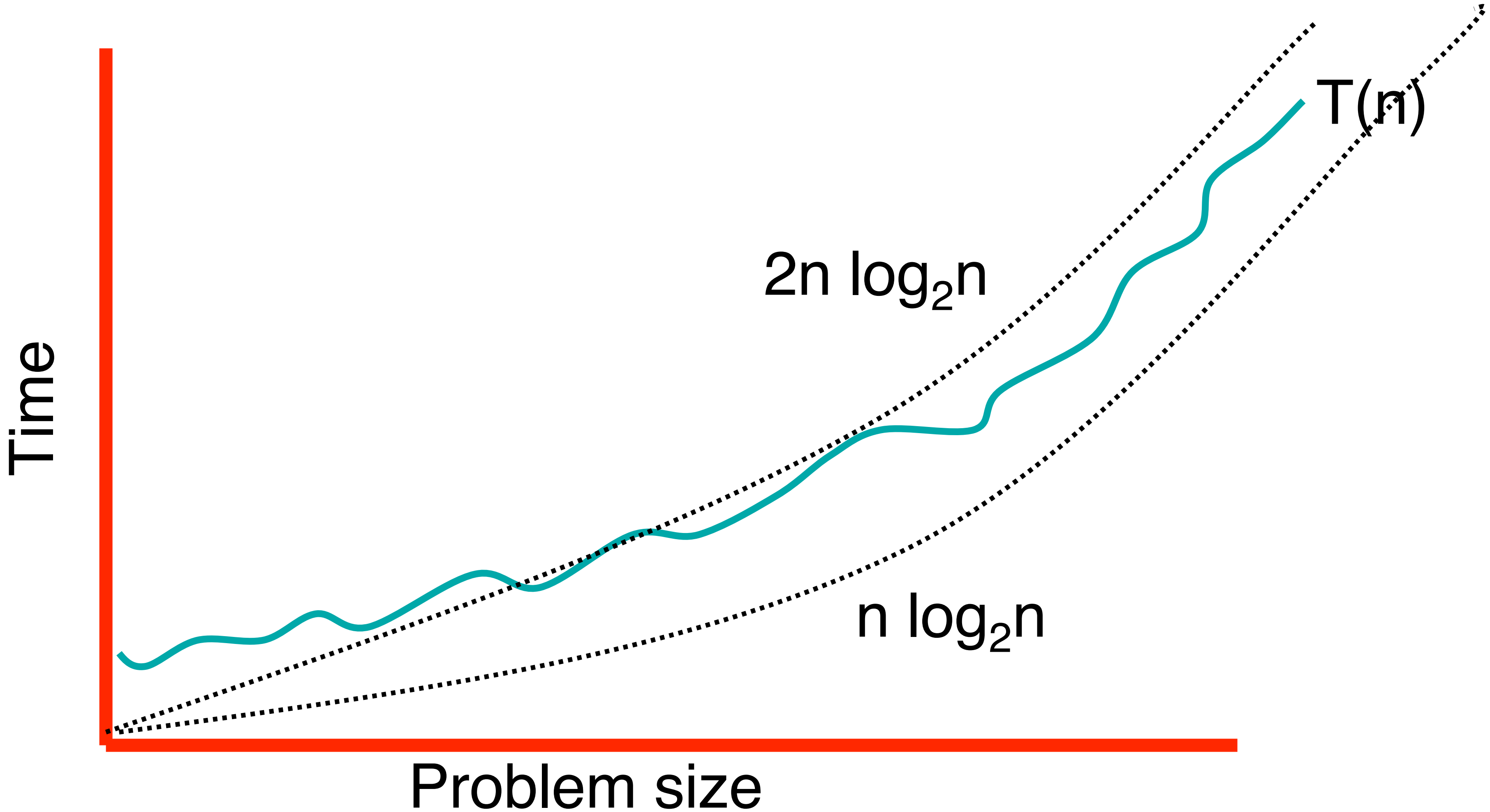
- The complexity of an algorithm associates a number **T(n)**, the best/worst/average-case time the algorithm takes, with each problem size **n**.

- Mathematically,
  - $\mathbf{T: N^+ \rightarrow R^+}$
  - that is **T** is a function that maps positive integers giving problem size to positive real numbers giving number of steps.

Complexity



Complexity



O-notation etc

- Given two functions **f** and **g:N→R**
  - **f(n)** is **O(g(n))** iff there is a constant **c**>0 so that **c g(n)** is eventually always ≥ **f(n)**
  - **f(n)** is **Ω(g(n))** iff there is a constant **c**>0 so that **c g(n)** is eventually always ≤ **f(n)**
  - **f(n)** is **Θ(g(n))** iff there is are constants c_1 and c_2>0 so that eventually always $c_1 g(n) \leq f(n) \leq c_2 g(n)$

Examples

- **$10n^2$-16n+100** is **$O(n^2)$** also **$O(n^3)$**
  - $10n^2-16n+100 \le 11n^2$ for all $n \ge 10$
- **$10n^2$-16n+100** is **$\Omega(n^2)$** also **$\Omega(n)$**
  - $10n^2-16n+100 \ge 9n^2$ for all $n \ge 16$
  - Therefore also **$10n^2$-16n+100** is **$\Theta(n^2)$**
- **$10n^2$-16n+100** is **not $O(n)$** also **not $\Omega(n^3)$**

“One-Way Equalities”

- “2 + 2 is 4” vs 2 + 2 = 4 vs 4 = 2 + 2
- “Every dog is a mammal” vs
  “Every mammal is a dog”
- $2n^2 + 5\ n$ is $O(n^3)$ vs
  $2n^2 + 5\ n = O(n^3)$ vs
  $O(n^3) = 2n^2 + 5\ n$ FALSE
- OK to put big-O in R.H.S. of equality, but not left; better to avoid both.

Domination

- **f(n)** is **o(g(n))** iff **$\lim_{n \to \infty}$ f(n)/g(n)=0**
  - that is **g(n) dominates f(n)**
- If $\alpha \leq \beta$ then n^{α} is $O(n^{\beta})$
- If $\alpha < \beta$ then n^{α} is $o(n^{\beta})$

- **Note:** if **f(n)** is **Θ(g(n))** then it cannot be **o(g(n))**

Working with O-Ω-Θ notation

- Claim: For any a, b>1 $\log_a n$ is $\Theta(\log_b n)$
  - $\log_a n = \log_a b \log_b n$ so letting $c = \log_a b$ we get that $c\log_b n \leq \log_a n \leq c\log_b n$

- Claim: For any a, and b>0, $(n+a)^b$ is $\Theta(n^b)$
  - $(n+a)^b \leq (2n)^b$ for $n \geq |a|$
    $= 2^b n^b = cn^b$ for $c = 2^b$ so $(n+a)^b$ is $O(n^b)$
  - $(n+a)^b \geq (n/2)^b$ for $n \geq 2|a|$ (even if a <0)
    $= 2^{-b} n^b = c'n$ for $c' = 2^{-b}$ so $(n+a)^b$ is $\Omega(n^b)$

Working with little-o

- $n^2 = o(n^3)$ [Use algebra]:

$$\lim_{n\to\infty} \frac{n^2}{n^3} = \lim_{n\to\infty} \frac{1}{n} = 0$$

- $n^3 = o(e^n)$ [Use L'Hospital's rule 3 times]:

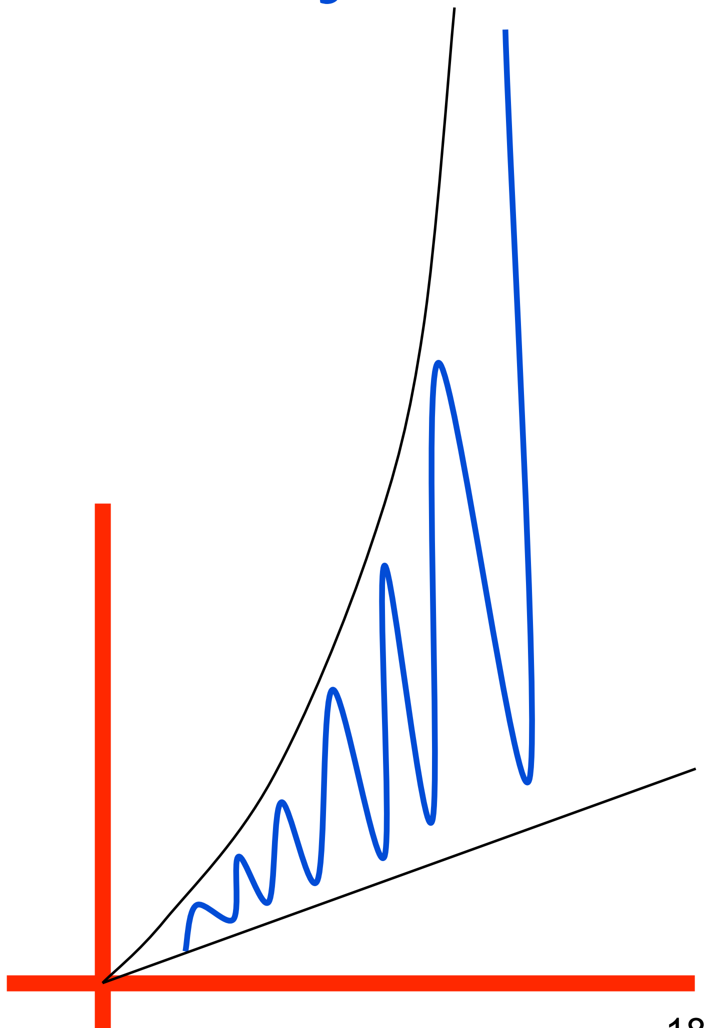
$$\lim_{n\to\infty} \frac{n^3}{e^n} = \lim_{n\to\infty} \frac{3n^2}{e^n} = \lim_{n\to\infty} \frac{6n}{e^n} = \lim_{n\to\infty} \frac{6}{e^n} = 0$$

Big-Theta, etc. not always "nice"

$$f(n) = \begin{cases} n^2, & n \text{ even} \\ n, & n \text{ odd} \end{cases}$$

$f(n) \neq \Theta(n^a)$ for any a.

Fortunately, such nasty cases are rare



A Possible Misunderstanding?

- We have looked at
  - type of complexity analysis
    - worst-, best-, average-case
  - types of function bounds
    - O, Ω, Θ
- These two considerations are independent of each other
  - one can do any type of function bound with any type of complexity analysis

Insertion Sort:

$\Omega(n^2)$ (worst case)

$O(n)$ (best case)