CSE 417: Algorithms and Computational Complexity

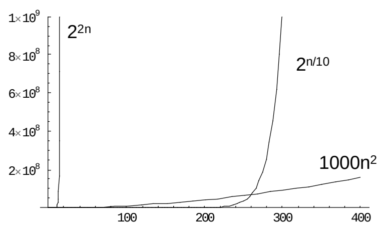
Winter 2002

P, NP, and Beyond

More History

- 1930's
  - – What is (is not) computable
- 1960/70's
  - – What is (is not) *feasibly* computable
  - Goal – a (largely) technology independent theory of time required by algorithms
  - Key modeling assumptions/approximations
    - Asymptotic (Big-O), worst case is revealing
    - Polynomial, exponential time – qualitatively different

Polynomial vs Exponential Growth



Another view of Poly vs Exp

Next year's computer will be 2x faster. If I can solve problem of size N_0 today, how large a problem can I solve in the same time next year?

Complexity	Increase	E.g. $T=10^{12}$	
O(n)	$n_0 \to 2n_0$	10^{12}	2×10^{12}
$O(n^2)$	$n_0 \to \sqrt{2}\, n_0$	10^6	1.4×10^6
$O(n^3)$	$n_0 \to \sqrt[3]{2}\, n_0$	10^4	1.25×10^4
$2^{n/10}$	$n_0 \to n_0+10$	400	410
2^n	$n_0 \to n_0+1$	40	41

Polynomial versus exponential

- We'll say any algorithm whose run-time is
  - polynomial is good
  - bigger than polynomial is bad
- Note – of course there are exceptions:
  - n^{100} is bigger than $(1.001)^n$ for most practical values of n but usually such run-times don't show up
  - There are algorithms that have run-times like $O(2^{n/22})$ and these may be useful for small input sizes, but they're not too common either

Some Convenient Technicalities

- "Problem" – the general case
  - Ex: The Clique Problem: Given a graph G and an integer k, does G contain a k-clique?
- "Problem Instance" – the specific cases
  - Ex: Does  contain a 4-clique? (no)
  - Ex: Does contain a 3-clique? (yes)
- Decision Problems – Just Yes/No answers
- Problems as Sets of "Yes" Instances
  - Ex: CLIQUE = { (G,k) | G contains a k-clique }

Decision problems

- Computational complexity usually analyzed using decision problems
  - answer is just 1 or 0 (yes or no).
- Why?
  - much simpler to deal with
  - *deciding* whether G has a k-clique, is certainly no harder than *finding* a k-clique in G, so a *lower* bound on deciding is also a lower bound on finding
  - Less important, but if you have a good decider, you can often use it to get a good finder. (Ex.: does G still have a k-clique after I remove this vertex?)

Computational Complexity

- Classify problems according to the amount of computational resources used by the best algorithms that solve them
- Recall:
  - worst-case running time of an algorithm
    - **max** # steps algorithm takes on any input of size n
- Define:
  - TIME(f(n)) to be the set of all decision problems solved by algorithms having worst-case running time O(f(n))

Polynomial time

- Define P (polynomial-time) to be
  - the set of all decision problems solvable by algorithms whose worst-case running time is bounded by some polynomial in the input size.
- $P = \bigcup_{k\geq 0} \mathrm{TIME}(n^k)$

Beyond P?

- There are many natural, practical problems for which we don't know any polynomial-time algorithms
- e.g. decisionTSP:
  - Given a weighted graph G and an integer k, does there exist a tour that visits all vertices in G having total weight at most k?

Solving TSP given a solution to decisionTSP

- Use binary search and several calls to decisionTSP to figure out what the exact total weight of the shortest tour is.
  - Upper and lower bounds to start are n times largest and smallest weights of edges, respectively
  - Call W the weight of the shortest tour.
- Now figure out which edges are in the tour
  - For each edge e in the graph in turn, remove e and see if there is a tour of weight at most W using decisionTSP
    - if not then e must be in the tour so put it back

More examples

- Independent-Set:
  - Given a graph G=(V,E) and an integer k, is there a subset U of V with $|U| \geq k$ such that no two vertices in U are joined by an edge.
- Clique:
  - Given a graph G=(V,E) and an integer k, is there a subset U of V with $|U| \geq k$ such that every pair of vertices in U is joined by an edge.

Satisfiability

- Boolean variables $x_1,\ldots,x_n$
  - taking values in {0,1}. 0=false, 1=true
- Literals
  - x_i or $\neg x_i$ for $i=1,\ldots,n$
- Clause
  - a logical OR of one or more literals
  - e.g. $(x_1 \vee \neg x_3 \vee x_7 \vee x_{12})$
- CNF formula
  - a logical AND of a bunch of clauses

13

Satisfiability

- CNF formula example
  - $(x_1 \vee \neg x_3 \vee x_7 \vee x_{12}) \wedge (x_2 \vee \neg x_4 \vee x_7 \vee x_5)$
- If there is some assignment of 0's and 1's to the variables that makes it true then we say the formula is satisfiable
  - the one above is, the following isn't
  - $x_1 \wedge (\neg x_1 \vee x_2) \wedge (\neg x_2 \vee x_3) \wedge \neg x_3$
- Satisfiability: Given a CNF formula F, is it satisfiable?

14

More History – As of 1970

- Many of the above problems had been studied for decades
- All had real, practical applications
- *None* had poly time algorithms; exponential was best known
- But, it turns out they all have a very deep similarity under the skin

15

Common property of these problems

- There is a special piece of information, a short hint or proof, that allows you to efficiently verify (in polynomial-time) that the YES answer is correct. This hint might be very hard to find
- e.g.
  - DecisionTSP: the tour itself,
  - Independent-Set, Clique: the set U
  - Satisfiability: an assignment that makes F true.

16

The complexity class NP

NP consists of all decision problems where

- You can verify the YES answers efficiently (in polynomial time) given a short (polynomial-size) hint

And

- No hint can fool your polynomial time verifier into saying YES for a NO instance

17

More Precise Definition of NP

- A decision problem is in NP iff there is a polynomial time procedure v(.,.), and an integer k such that
  - for every YES problem instance x there is a hint h with $|h| \leq |x|^k$ such that v(x,h) = YES

  and
  - for every NO problem instance x there is *no* hint h with $|h| \leq |x|^k$ such that v(x,h) = YES

18

Example: CLIQUE is in NP

procedure v(x,h)
- if
  - x is a well-formed representation of a graph G = (V, E) and an integer k,
- and
  - h is a well-formed representation of a k vertex subset U of V,
- and
  - U is a clique in G,
- then output "YES"
- else output "I'm unconvinced"

Is it correct?

- For every x = (G,k) such that G contains a k-clique, there is a hint h that will cause v(x,h) to say YES, namely h = a list of the vertices in such a k-clique

and

- No hint can fool v into saying yes if either x isn't well-formed (the uninteresting case) or if x = (G,k) but G does not have any cliques of size k (the interesting case)

Keys to showing that a problem is in NP

- What's the output? (must be YES/NO)
- What's the input? Which are YES?
- For every given YES input, is there a hint that would help?
  - OK if some inputs need no hint
- For any given NO input, is there a hint that would trick you?

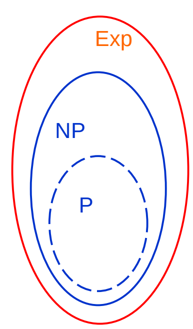
Solving NP problems without hints

- The only obvious algorithm for most of these problems is brute force:
  - try all possible hints and check each one to see if it works.
  - *Exponential* time:
    - 2^n truth assignments for n variables
    - n! possible TSP tours of n vertices
    - $\binom{n}{k}$ possible k element subsets of n vertices
    - etc.

What We Know

- Nobody knows if all problems in NP can be done in polynomial time, i.e. does P=NP?
  - one of the most important open questions in all of science.
  - huge practical implications
- Every problem in P is in NP
  - one doesn't even need a hint for problems in P so just ignore any hint you are given
- Every problem in NP is in exponential time

P and NP



P vs NP

- Theory
  - P = NP ?
  - Open Problem!
  - I bet against it
- Practice
  - Many interesting, useful, natural, well-studied problems known to be NP-complete
  - With rare exceptions, no one routinely succeeds in finding exact solutions to large, arbitrary instances

25

More Connections

- Some Examples in NP
  - Satisfiability
  - Independent-Set
  - Clique
  - Vertex Cover
- All hard to solve; hints seem to help on all
- Very surprising fact:
  - Fast solution to *any* gives fast solution to *all!*

26

NP-hardness & NP-completeness

- Some problems in NP seem hard
  - people have looked for efficient algorithms for them for hundreds of years without success
- However
  - nobody knows how to **prove** that they are really hard to solve, i.e. $P \neq NP$

27

NP-hardness & NP-completeness

- Alternative approach
  - show that they are at least as hard as any problem in NP
- Rough definition:
  - A problem is NP-hard iff it is at least as hard as any problem in NP
  - A problem is NP-complete iff it is both
    - NP-hard
    - in NP

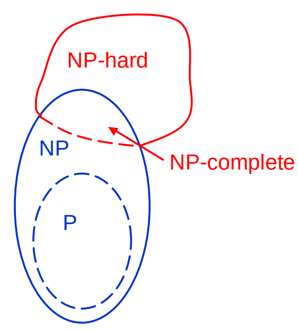
28

Polynomial Time Reduction

- $L \le^p R$ if there is a poly time algorithm for L *assuming* a poly time subroutine for R
- Thus, fast alg for R implies fast alg for L
- If you can prove there is no fast alg for L, then that proves there is no fast alg for R

29

P and NP



30

What to do? Hopeless?

- Heuristics: perhaps there's an alg that's:
  - usually fast, and/or
  - usually succeeds
- Approximation Algorithms: Would you settle for an answer within 1% of optimal? 10% ? 10x ?

31

Is NP as bad as it gets?

- NO! NP-complete problems are frequently encountered, but there's worse:
  - Some problems provably require exponential time.
    - Ex: Does P halt on x in $2^{|x|}$ steps?
  - Some require 2^n, 2^{2^n}, $2^{2^{2^n}}$, ... steps
  - And of course, some are just plain uncomputable

32

Summary

- Big-O – good
- P – good
- Exp – bad
- Hints help? NP
- NP-hard, NP-complete – bad (I bet)

33