CSE 417: Algorithms and Computational Complexity

Winter 2002

Graphs and Graph Algorithms

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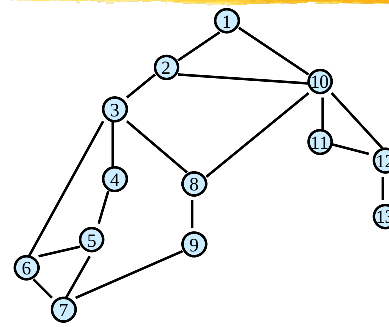
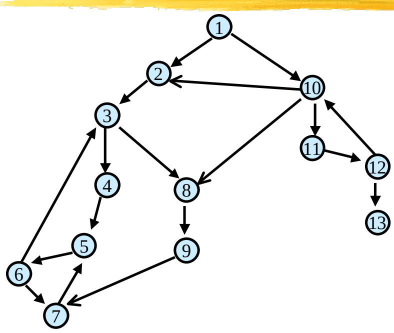
Undirected Graph G = (V,E)



Directed Graph G = (V,E)

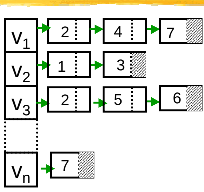


Representing Graph G=(V,E) n vertices, m edges

- Vertex set $V=\{v_1,\ldots v_n\}$
- Adjacency Matrix A
  - $A[i,j]=1$ iff $(v_i,v_j) \in E$
  - Space is n^2 bits
- Advantages:
  - O(1) test for presence or absence of edges.
  - compact in packed binary form for large m
- Disadvantages: inefficient for sparse graphs

Representing Graph G=(V,E) n vertices, m edges

- Adjacency List:
  - O(n+m) words
  - O(log n) bits each



- Advantages:
  - Compact for sparse graphs

Representing Graph G=(V,E) n vertices, medges

- Adjacency List:
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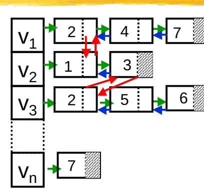


- Back- and cross pointers more work to build, but allow easier traversal and deletion of edges
  - usually assume this format

Graph Traversal

- Learn the basic structure of a graph
- Walk from a fixed starting vertex v to find all vertices reachable from v
- Three states of vertices
  - **undiscovered**
  - **discovered**
  - **fully-explored**

Breadth-First Search

- Completely explore the vertices in order of their distance from v
- Naturally implemented using a queue

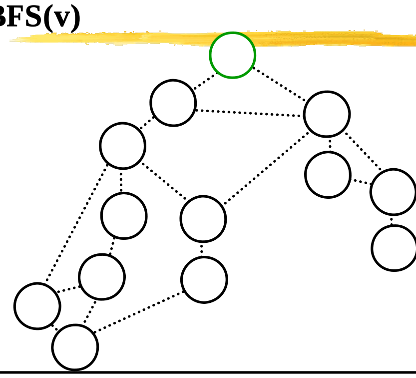
BFS(v)

Global initialization: mark all vertices "undiscovered"

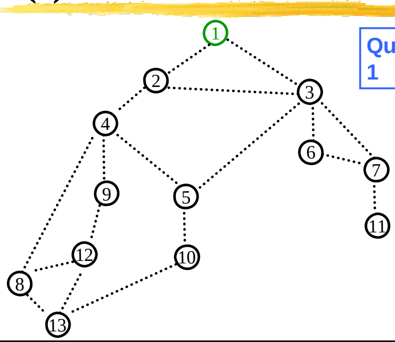
```
BFS(v)
    mark  v "discovered"
    queue = v
    while queue not empty
        u = remove_first(queue)
        for each edge {u,x}
            if (x is undiscovered)
                mark x discovered
                append x on queue
        mark u completed
```

Exercise: modify code to number vertices & compute level numbers

BFS(v)

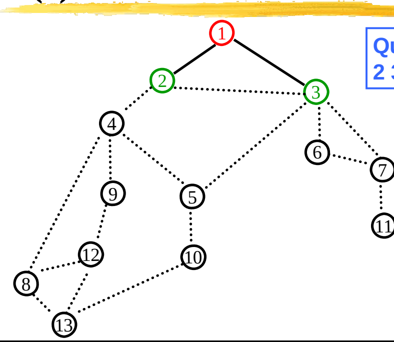


BFS(v)



Queue:
1

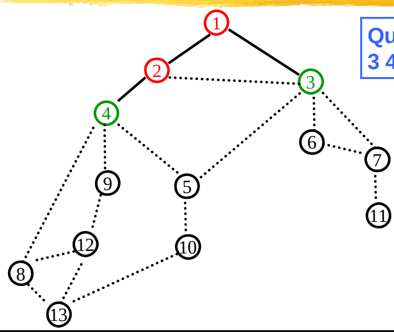
BFS(v)



Queue:
2 3

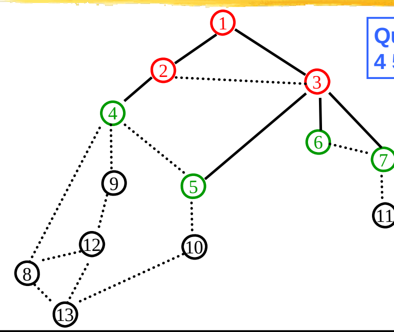
BFS(v)

Queue:
3 4



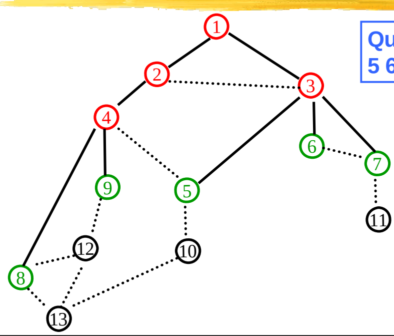
BFS(v)

Queue:
4 5 6 7



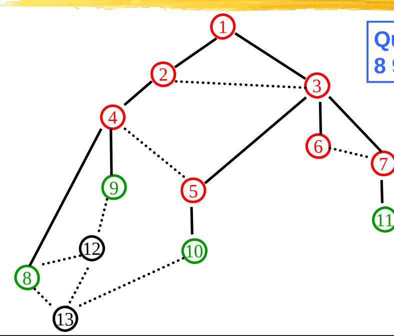
BFS(v)

Queue:
5 6 7 8 9



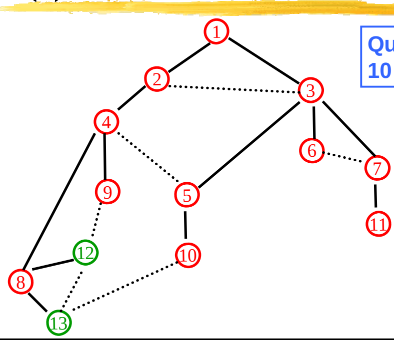
BFS(v)

Queue:
8 9 10 11



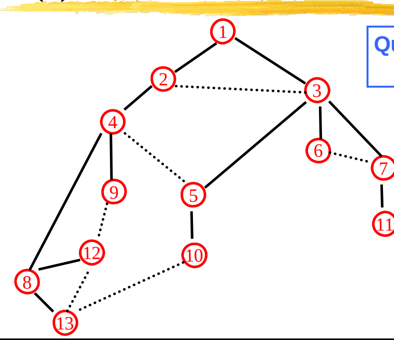
BFS(v)

Queue:
10 11 12 13



BFS(v)

Queue:



BFS analysis

- Each edge is explored once from each end-point (at most)
- Each vertex is discovered by following a different edge
- Total cost O(m) where m=# of edges

Properties of (Undirected) BFS(v)

- BFS(v) visits x if and only if there is a path in G from v to x.
- Edges into then-undiscovered vertices define a ***tree*** – the "breadth first spanning tree" of G
- Level i in this tree are exactly those vertices u such that the shortest path (in G, not just the tree) from the root v is of length i.
- ***All*** non-tree edges join vertices on the same or adjacent levels

Graph Search Application: Connected Components

- Want to answer questions of the form:
  - given vertices u and v, is there a path from u to v?
- Idea: create array A such that
  A[u] = smallest numbered vertex that is connected to u
- question reduces to whether A[u]=A[v]?

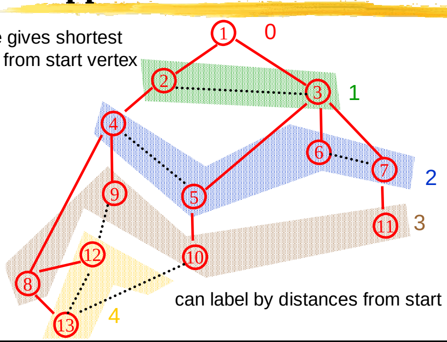
Q: Why not create 2-d array Path[u,v]?

Graph Search Application: Connected Components

- initial state: all v undiscovered
  for v=1 **to** n **do**
  if state(v)!=fully-explored **then**
  BFS(v): **setting** A[u] ←v **for each** u **found**
  (and marking u discovered/fully-explored)
  endif
  endfor
- Total cost: O(n+m)
  - each vertex an each edge is touched a constant number of times
  - works also with DFS

BFS Application: Shortest Paths

Tree gives shortest paths from start vertex



can label by distances from start

Depth-First Search

- Follow the first path you find as far as you can go
- Back up to last unexplored edge when you reach a dead end, then go as far you can
- Naturally implemented using recursive calls or a stack

DFS(v)

Global Initialization: mark all vertices "undiscovered"

DFS(v)
    mark v "discovered"
    stack = v
    while stack not empty
        u = pop(stack)
        for each edge {u,x}
            if (x is undiscovered)
                mark x discovered
                push x
        mark u completed

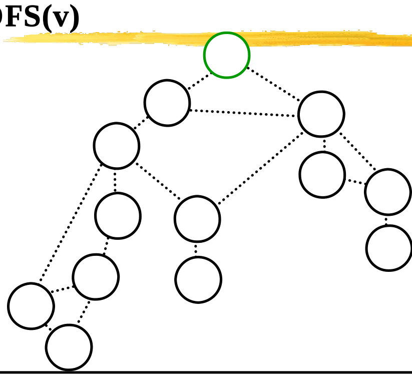
> Exercise1: recode recursively
>
> Exercise 2: modify to compute vertex numbering

DFS(v) – Recursive version

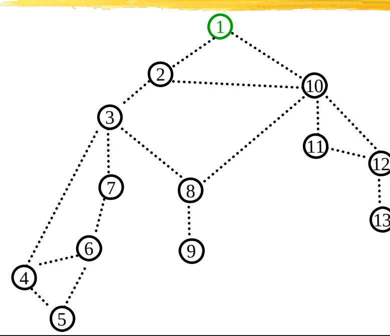
Global Initialization:
    mark all vertices v "undiscovered" via v.dfs# = -1
    dfscounter = 0

DFS(v)
    v.dfs# = dfscounter++   // mark v "discovered"
    for each edge (v,x)
        if (x.dfs# = -1)   // tree edge (x previously undiscovered)
            DFS(x)
        else …   // code for back-, fwd-, parent,
                                        // edges, if needed
    // mark v "completed," if needed

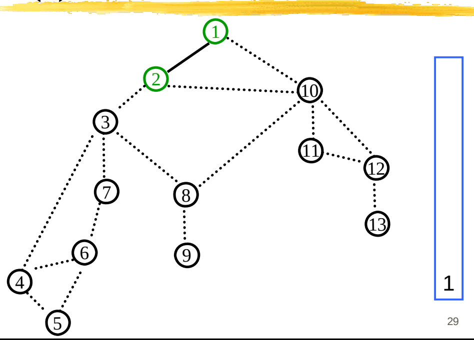
DFS(v)



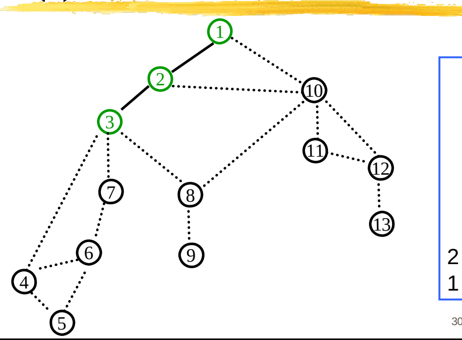
DFS(v)



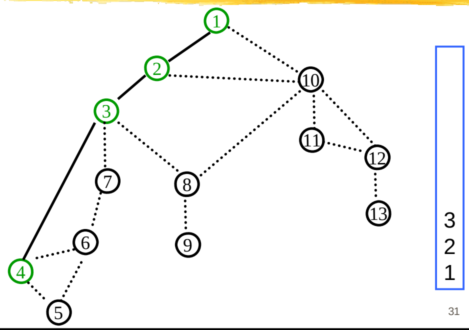
DFS(v)



DFS(v)



DFS(v)



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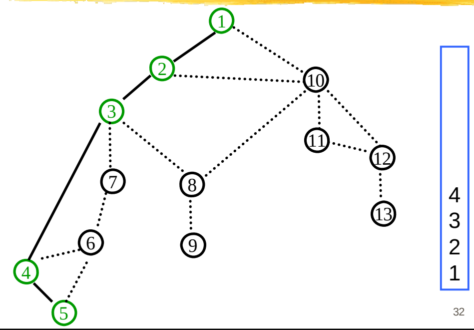
DFS(v)



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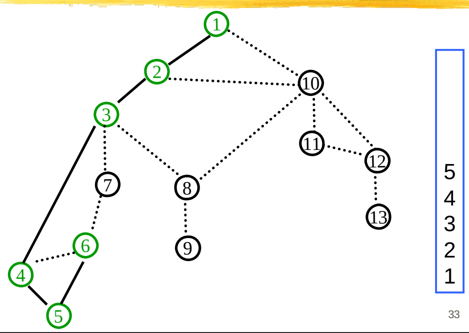
DFS(v)



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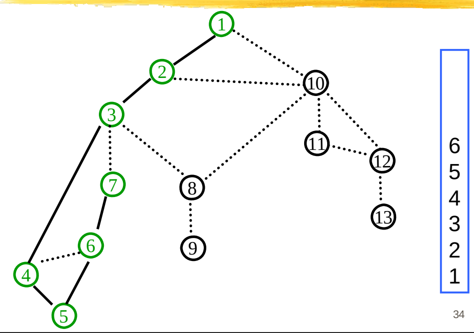
DFS(v)



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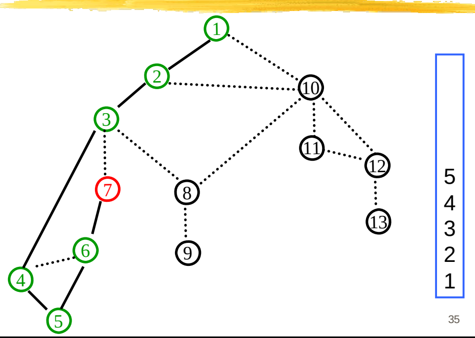
DFS(v)



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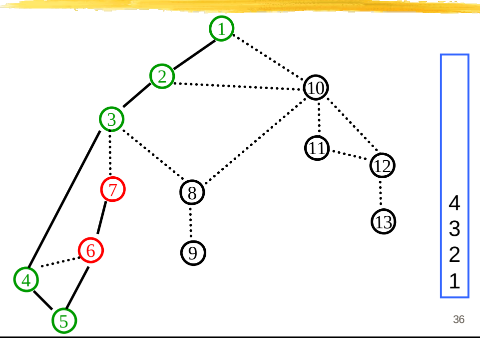
DFS(v)



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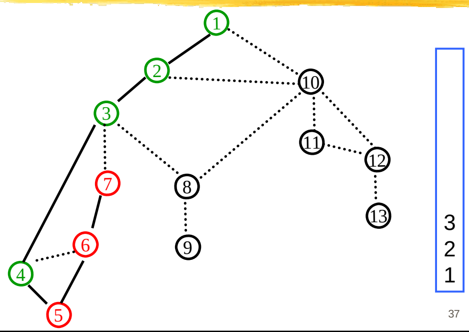
DFS(v)



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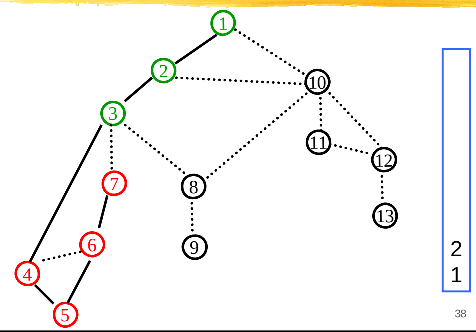
DFS(v)



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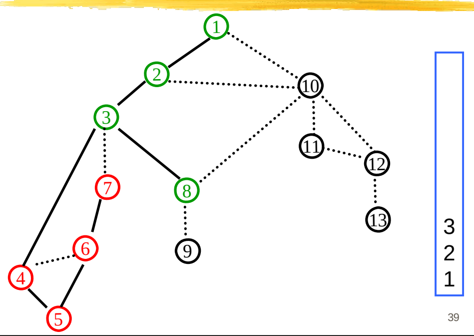
DFS(v)



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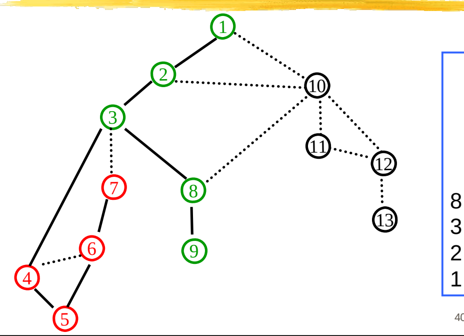
DFS(v)



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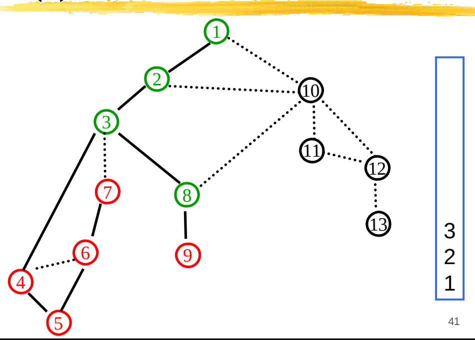
DFS(v)



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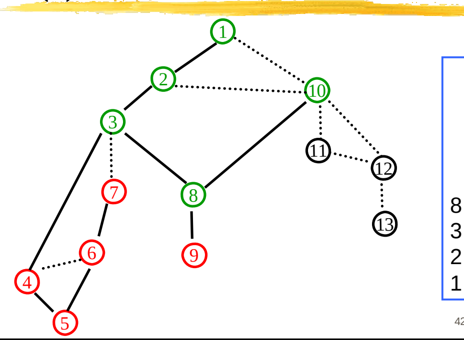
DFS(v)



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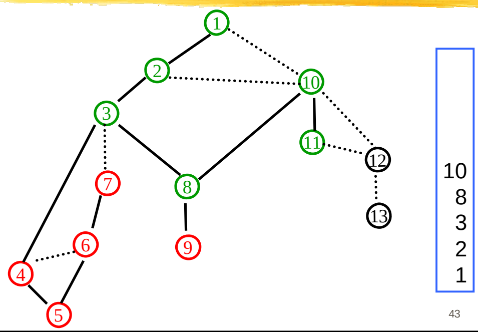
DFS(v)



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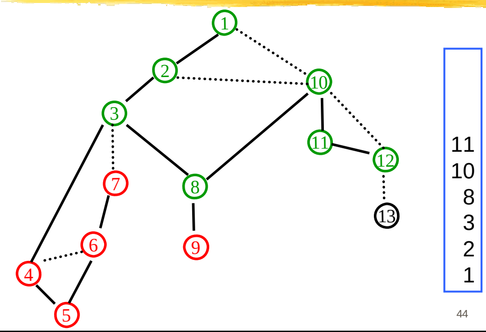
DFS(v)



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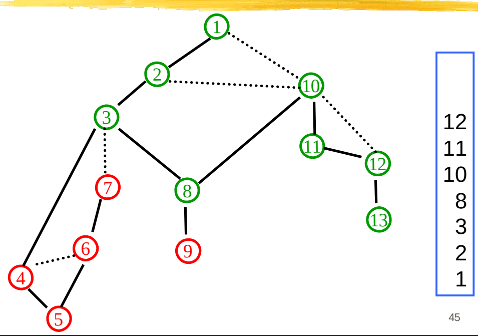
DFS(v)



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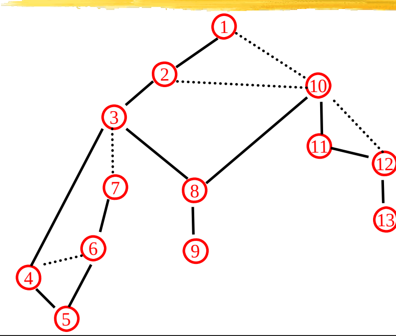
DFS(v)



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Properties of (Undirected) DFS(v)

- Like BFS(v):
  - DFS(v) visits x if and only if there is a path in G from v to x (through previously unvisited vertices)
  - Edges into then-undiscovered vertices define a ***tree*** – the "depth first spanning tree" of G
- Unlike the BFS tree:
  - the DF spanning tree isn't minimum depth
  - its levels don't reflect min distance from the root
  - non-tree edges never join vertices on the same or adjacent levels
- BUT…

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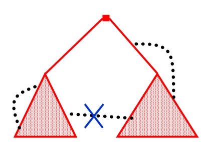
Non-tree edges

- All non-tree edges join a vertex and one of its descendents/ancestors in the DFS tree
- No cross edges!



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Application: Articulation Points

- A node in an undirected graph is an **articulation point** iff removing it disconnects the graph
- articulation points represent vulnerabilities in a network – single points whose failure would split the network into 2 or more disconnected components

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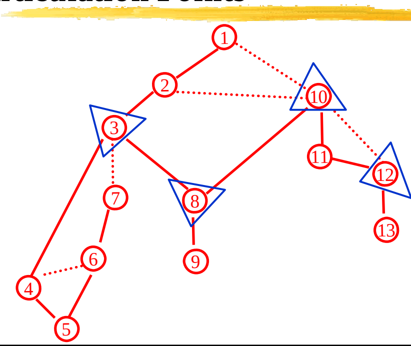
Articulation Points



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Brainstorming

- draw a graph, ~ 10 nodes, A-J
- redraw as via DFS
- add dsf#s & tree/back edges (solid/dashed)
- find cycles
- give alg to find cycles via dfs; does G have any?

- find articulation points
- what do cycles have to do with articulation points?
- alg to find articulation points via DFS???

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Articulation Points from DFS

- Every interior vertex of a tree is an articulation point
  - Non-tree edges eliminate articulation points
- Root node is an articulation point iff it has more than one child

non-leaf, non-root node u is an articulation point ⟺ no non-tree edge goes above u from a sub-tree below some child of u

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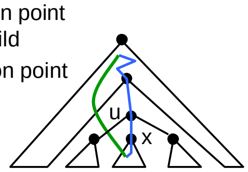
Articulation Points from DFS

- Root node is an articulation point iff it has more than one child
- Leaf is never an articulation point
- non-leaf, non-root node u is an articulation point

⇕

no non-tree edge goes above u from a sub-tree below some child of u



If removal of u does NOT separate x, there must be an exit from x's subtree. How? Via back edge.

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Articulation Points: the "LOW" function

- Definition: LOW(v) is the lowest dfs# of any vertex that is either in the dfs subtree rooted at v (including v itself) or connected to a vertex in that subtree by a back edge.
- Key idea 1: if some child x of v has LOW(x) ≥ dfs#(v) then v is an articulation point.
- Key idea 2: LOW(v) =
  min ({LOW(w) | w a child of v } ∪
  { dfs#(x) | {v,x} is a back edge from v })

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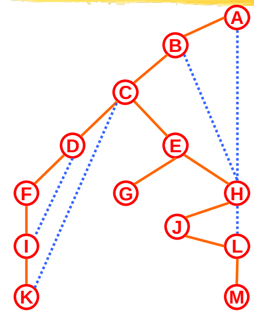
DFS(v) for Finding Articulation Points

```
Global initialization: v.dfs# = -1 for all v.
DFS(v)
 v.dfs# = dfscounter++
 v.low = v.dfs#              // initialization
 for each edge {v,x}
    if (x.dfs# == -1)        // x is undiscovered
       DFS(x)
       v.low = min(v.low, x.low)
       if (x.low >= v.dfs#)
          print "v is art. pt., separating x"
    else if (x is not v's parent)
       v.low = min(v.low, x.dfs#)
```

Except for root. Why?

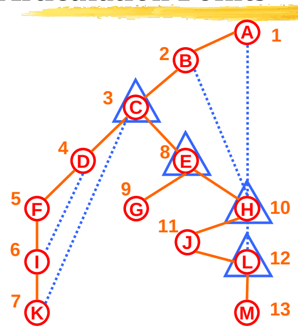
Equiv: "if({v,x} is a back edge)" Why?

Articulation Points



Vertex	DFS #	Low
A		
B		
C		
D		
E		
F		
G		
H		
I		
J		
K		
L		
M		

Articulation Points



Vertex	DFS #	Low
A	1	1
B	2	1
C	3	1
D	4	3
E	8	1
F	5	3
G	9	9
H	10	1
I	6	3
J	11	10
K	7	3
L	12	10
M	13	13