CSE 417: Algorithms and Computational Complexity

Divide & Conquer II

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Sometimes two sub-problems aren't enough

- More general divide and conquer
  - You've broken the problem into **a** different sub-problems
  - Each has size at most **n/b**
  - The cost of the break-up and recombining the sub-problem solutions is **O(n^k)**
- Recurrence
  - $T(n)= a \cdot T(n/b)+c \cdot n^k$

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Master Divide and Conquer Recurrence

- If $T(n)= a \cdot T(n/b)+c \cdot n^k$ for $n>b$ then
  - if $a>b^k$ then $T(n)$ is $Q(n^{\log_b a})$
  - if $a<b^k$ then $T(n)$ is $Q(n^k)$
  - if $a=b^k$ then $T(n)$ is $Q(n^k \log n)$
- Works even if it is $\lceil n/b \rceil$ instead of n/b.

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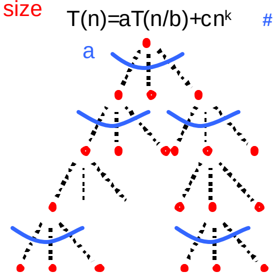
Proving Master recurrence



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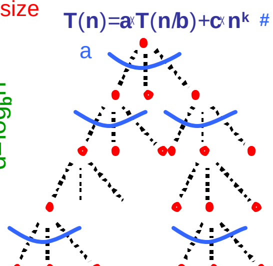
Proving Master recurrence



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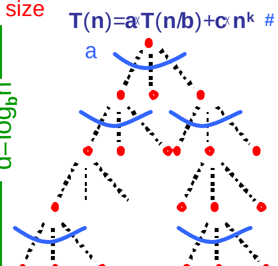
Proving Master recurrence



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Geometric Series

- $S = t + tr + tr^2 + \ldots + tr^{n-1}$
- $rS = tr + tr^2 + \ldots + tr^{n-1} + tr^n$
- $(r-1)S = tr^n - t$
- so $S = t(r^n - 1)/(r-1)$ if $r?1$.

- Simple rule
  - If $r ? 1$ then S is a constant times largest term in series

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Total Cost

- Geometric series
  - ratio a/b^k
  - $d+1 = \log_b n + 1$ terms
  - first term cn^k, last term ca^d
- If $a/b^k = 1$
  - all terms are equal $T(n)$ is $Q(n^k \log n)$
- If $a/b^k < 1$
  - first term is largest $T(n)$ is $\Theta(n^k)$
- If $a/b^k > 1$
  - last term is largest $T(n)$ is $\Theta(a^d) = \Theta(a^{\log_b n}) = \Theta(n^{\log_b a})$
  (To see this take $\log_b$ of both sides)

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Multiplying Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \bullet \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}+a_{14}b_{42} & \circ & a_{11}b_{14}+a_{12}b_{24}+a_{13}b_{34}+a_{14}b_{44} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}+a_{24}b_{41} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}+a_{24}b_{42} & \circ & a_{21}b_{14}+a_{22}b_{24}+a_{23}b_{34}+a_{24}b_{44} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}+a_{34}b_{41} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}+a_{34}b_{42} & \circ & a_{31}b_{14}+a_{32}b_{24}+a_{33}b_{34}+a_{34}b_{44} \\ a_{41}b_{11}+a_{42}b_{21}+a_{43}b_{31}+a_{44}b_{41} & a_{41}b_{12}+a_{42}b_{22}+a_{43}b_{32}+a_{44}b_{42} & \circ & a_{41}b_{14}+a_{42}b_{24}+a_{43}b_{34}+a_{44}b_{44} \end{bmatrix}$$

- n^3 multiplications, $n^3 - n^2$ additions

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Multiplying Matrices

```
for i=1 to n
    for j=1 to n
        C[i,j]←0
        for k=1 to n
            C[i,j]=C[i,j]+A[i,k]·B[k,j]
        endfor
    endfor
endfor
```

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Multiplying Matrices

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Multiplying Matrices

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(blocks labelled A_{11}, A_{12}, A_{21}, A_{22} and B_{11}, B_{12}, B_{21}, B_{22})

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}+a_{14}b_{42} & \circ & a_{11}b_{14}+a_{12}b_{24}+a_{13}b_{34}+a_{14}b_{44} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}+a_{24}b_{41} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}+a_{24}b_{42} & \circ & a_{21}b_{14}+a_{22}b_{24}+a_{23}b_{34}+a_{24}b_{44} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}+a_{34}b_{41} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}+a_{34}b_{42} & \circ & a_{31}b_{14}+a_{32}b_{24}+a_{33}b_{34}+a_{34}b_{44} \\ a_{41}b_{11}+a_{42}b_{21}+a_{43}b_{31}+a_{44}b_{41} & a_{41}b_{12}+a_{42}b_{22}+a_{43}b_{32}+a_{44}b_{42} & \circ & a_{41}b_{14}+a_{42}b_{24}+a_{43}b_{34}+a_{44}b_{44} \end{bmatrix}$$

(blocks labelled $A_{11}B_{11}+A_{12}B_{21}$, $A_{11}B_{12}+A_{12}B_{22}$, $A_{21}B_{11}+A_{22}B_{21}$, $A_{21}B_{12}+A_{22}B_{22}$)

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Simple Divide and Conquer

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{bmatrix}$$

- $T(n)=8T(n/2)+4(n/2)^2=8T(n/2)+n^2$
  - $8>2^2$ so $T(n)$ is $Q(n^{\log_b a}) = Q(n^{\log_2 8}) = Q(n^3)$

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Strassen's Divide and Conquer Algorithm

- Strassen's algorithm
  - Multiply 2×2 matrices using 7 instead of 8 multiplications (and lots more than 4 additions)
  - $T(n)=7\ T(n/2)+cn^2$
    - $7>2^2$ so $T(n)$ is $Q(n^{\log_2 7})$ which is $O(n^{2.81...})$
  - Fastest algorithms theoretically use $O(n^{2.376})$ time
    - not practical but Strassen's is practical **provided calculations are exact** and we stop recursion when matrix has size about 100 (maybe 10)

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The algorithm

$P_1 \leftarrow A_{12}(B_{11}+B_{21})$; $P_2 \leftarrow A_{21}(B_{12}+B_{22})$

$P_3 \leftarrow (A_{11} - A_{12})B_{11}$; $P_4 \leftarrow (A_{22} - A_{21})B_{22}$

$P_5 \leftarrow (A_{22} - A_{12})(B_{21} - B_{22})$

$P_6 \leftarrow (A_{11} - A_{21})(B_{12} - B_{11})$

$P_7 \leftarrow (A_{21} - A_{12})(B_{11}+B_{22})$

$C_{11} \leftarrow P_1+P_3$; $C_{12} \leftarrow P_2+P_3+P_6 - P_7$

$C_{21} \leftarrow P_1+P_4+P_5+P_7$; $C_{22} \leftarrow P_2+P_4$

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Another Divide &Conquer Example: Multiplying Faster

- On the first HW you analyzed our usual algorithm for multiplying numbers
  - $Q(n^2)$ time
- We can do better!
  - We'll describe the basic ideas by multiplying polynomials rather than integers
  - Advantage is we don't get confused by worrying about carries at first

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Notes on Polynomials

- These are just formal sequences of coefficients
  - when we show something multiplied by x^k it just means shifted k places to the left – basically no work

Usual polynomial multiplication

$$\begin{array}{r} 3x^2 + 2x + 2 \\ x^2 - 3x + 1 \\ \hline 3x^2 + 2x + 2 \\ -9x^3 - 6x^2 - 6x \\ 3x^4 + 2x^3 + 2x^2 \\ \hline 3x^4 - 7x^3 - x^2 - 4x + 2 \end{array}$$

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Polynomial Multiplication

- Given:
  - Degree $n-1$ polynomials P and Q
    - $P = a_0 + a_1 x + a_2 x^2 + \ldots + a_{m-2}x^{n-2} + a_{m-1}x^{n-1}$
    - $Q = b_0 + b_1 x + b_2 x^2 + \ldots + b_{m-2}x^{n-2} + b_{m-1}x^{n-1}$
- Compute:
  - Degree $2n-2$ Polynomial PQ
  - $PQ = a_0b_0 + (a_0b_1+a_1b_0)\,x + (a_0b_2+a_1b_1+a_2b_0)\,x^2 + \ldots + (a_{n-2}b_{n-1}+a_{n-1}b_{n-2})\,x^{2n-3} + a_{n-1}b_{n-1}\,x^{2n-2}$
- Obvious Algorithm:
  - Compute all a_ib_j and collect terms
  - $Q(n^2)$ time

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Naive Divide and Conquer

- Assume $n=2k$
  - $P = (a_0 + a_1 x + a_2 x^2 + \ldots + a_{k-2} x^{k-2} + a_{k-1} x^{k-1}) + (a_k + a_{k+1} x + \ldots + a_{m-2}x^{k-2} + a_{m-1}x^{k-1})\, x^k$
    $= P_0 + P_1 x^k$ where P_0 and P_1 are degree $k-1$ polynomials
  - Similarly $Q = Q_0 + Q_1 x^k$
- $PQ = (P_0+P_1x^k)(Q_0+Q_1x^k)$
  $= P_0Q_0 + (P_1Q_0+P_0Q_1)x^k + P_1Q_1x^{2k}$
- 4 sub-problems of size $k=n/2$ plus linear combining
  - $T(n)=4T(n/2)+cn$ Solution $T(n) = Q(n^2)$

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