## CSE 417: Algorithms and Computational Complexity

Winter 2001

Lecture 5

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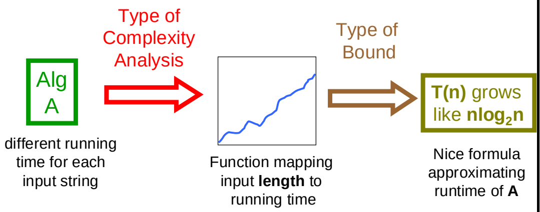
## Fixing a Misunderstanding

- We have looked at
  - type of complexity analysis
    - worst-case, best-case, average-case
  - types of function bounds
    - O, Ω, Θ
- These two considerations are orthogonal to each other
  - one can do any type of function bound with any type of complexity analysis



## Complexity analysis overview




Usually we represent the function in the middle using a recurrence relation rather than explicitly



## Quicksort Analysis

- Partition does **n-1** comparisons on a list of length **n**
  - pivot is compared to each other element
- If **pivot** is $i^{th}$ largest then two subproblems are of size **i-1** and **n-i**
- Pivot is equally likely to be any one of $1^{st}$ through $n^{th}$ largest

$$T(n)=n-1+\frac{1}{n}\sum_{i=1}^{n}(T(i-1)+T(n-i))$$



## Quicksort analysis

$$T(n)=n-1+\frac{1}{n}\sum_{i=1}^{n}(T(i-1)+T(n-i))$$

$$=n-1+\frac{2T(1)+2T(2)+\ldots+2T(n-1)}{n}$$

$$\therefore nT(n)=n(n-1)+2T(1)+2T(2)+\ldots+2T(n-1)$$

$$\underline{(n+1)T(n+1)=(n+1)n+2T(1)+2T(2)+\ldots+2T(n)}$$

$$\therefore (n+1)T(n+1)-nT(n)=2T(n)+2n$$

$$(n+1)T(n+1)=(n+2)T(n)+2n$$

$$\therefore \frac{T(n+1)}{n+2}=\frac{T(n)}{n+1}+\frac{2n}{(n+1)(n+2)}$$



## Quicksort analysis

$$\text{Let } Q(n)=\frac{T(n)}{n+1}$$

$$\therefore Q(n+1)\le Q(n)+\frac{2}{n+1}$$

$$\therefore Q(n)\le 2(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n})=2H_n\approx 2\ln n=1.38\log_2 n$$

(Recall that $\ln n=\int_1^n 1/x\ dx$)

$$\therefore T(n)\approx 1.38\, n\log_2 n$$

## Algorithm Design Techniques

- General overall idea
  - Reduce solving a problem to a smaller problem or problems of the same type
- Greedy algorithms
  - Used when one needs to build something a piece at a time
  - Repeatedly make the **greedy** choice - the one that looks the best right away
    - e.g. closest pair in TSP search
  - Usually fast if they work

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## Algorithm Design Techniques

- Divide & Conquer
  - Reduce problem to one or more sub-problems of the same type
  - Each sub-problem is at most a constant fraction of the size of the original problem
    - e.g. Mergesort, Binary Search, Strassen's Algorithm, Quicksort (kind of)

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## Fast exponentiation

- Power(a,n)
  - **Input:** integer **n** and number **a**
  - **Output:** $a^n$
- Obvious algorithm
  - **n-1** multiplications
- Observation:
  - if **n** is even, **n=2m**, then $a^n = a^m \bullet a^m$

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## Divide & Conquer Algorithm

- Power(a,n)

```
    if n=0 then
        return(1)
    else
        x ←Power(a,⌊n/2⌋)
        if n is even then
            return(x•x)
        else
            return(a•x•x)
```

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## Analysis

- Worst-case recurrence
  - $T(n)=T(\lfloor n/2 \rfloor)+2$
- By master theorem
  - $T(n)=O(\log n)$
- More precise analysis:
  - $T(n)= \lceil \log_2 n \rceil$ + # of 1's in n's binary representation

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## A Practical Application- RSA

- Instead of $a^n$ want $a^n$ **mod N**
  - $a^{i+j}$ mod **N** = (($a^i$ mod **N**)•($a^j$ mod **N**)) mod **N**
  - same algorithm applies with each **x•y** replaced by
    - ((**x** mod **N**)•(**y** mod **N**)) mod **N**
- In RSA cryptosystem (widely used for security)
  - need $a^n$ **mod N** where **a**, **n**, **N** each typically have 1024 bits
  - Power: at most 2048 multiplies of 1024 bit numbers
    - relatively easy for modern machines
  - Naive algorithm: $2^{1024}$ multiplies

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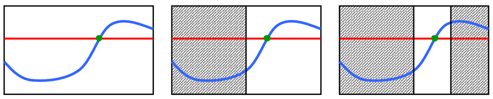
## Binary search for roots (bisection method)



- Given:
  - continuous function **f** and two points **a<b** with **f(a)<0** and **f(b)>0**
- Find:
  - approximation to **c** s.t. **f(c)=0** and **a<c<b**

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