CSE 417: Algorithms and Computational Complexity

Winter 2001

Lecture 4

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Master Divide and Conquer Recurrence

- If $T(n)=aT(n/b)+cn^k$ for $n>b$ then
  - if $a>b^k$ then $T(n)$ is $\Theta(n^{\log_b a})$
  - if $a<b^k$ then $T(n)$ is $\Theta(n^k)$
  - if $a=b^k$ then $T(n)$ is $\Theta(n^k \log n)$
- Works even if it is $\lceil n/b \rceil$ instead of n/b.

Multiplying Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \bullet \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}+a_{14}b_{42} & \circ & a_{11}b_{14}+a_{12}b_{24}+a_{13}b_{34}+a_{14}b_{44} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}+a_{24}b_{41} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}+a_{24}b_{42} & \circ & a_{21}b_{14}+a_{22}b_{24}+a_{23}b_{34}+a_{24}b_{44} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}+a_{34}b_{41} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}+a_{34}b_{42} & \circ & a_{31}b_{14}+a_{32}b_{24}+a_{33}b_{34}+a_{34}b_{44} \\ a_{41}b_{11}+a_{42}b_{21}+a_{43}b_{31}+a_{44}b_{41} & a_{41}b_{12}+a_{42}b_{22}+a_{43}b_{32}+a_{44}b_{42} & \circ & a_{41}b_{14}+a_{42}b_{24}+a_{43}b_{34}+a_{44}b_{44} \end{bmatrix}$$

- n^3 multiplications, n^3-n^2 additions

Multiplying Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \bullet \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

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Multiplying Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \bullet \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}+a_{14}b_{42} & \circ & a_{11}b_{14}+a_{12}b_{24}+a_{13}b_{34}+a_{14}b_{44} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}+a_{24}b_{41} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}+a_{24}b_{42} & \circ & a_{21}b_{14}+a_{22}b_{24}+a_{23}b_{34}+a_{24}b_{44} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}+a_{34}b_{41} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}+a_{34}b_{42} & \circ & a_{31}b_{14}+a_{32}b_{24}+a_{33}b_{34}+a_{34}b_{44} \\ a_{41}b_{11}+a_{42}b_{21}+a_{43}b_{31}+a_{44}b_{41} & a_{41}b_{12}+a_{42}b_{22}+a_{43}b_{32}+a_{44}b_{42} & \circ & a_{41}b_{14}+a_{42}b_{24}+a_{43}b_{34}+a_{44}b_{44} \end{bmatrix}$$

Multiplying Matrices

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \bullet \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{bmatrix}$$

Multiplying Matrices

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{pmatrix}$$

- $T(n)=8T(n/2)+4(n/2)^2=8T(n/2)+n^2$
  - $8>2^2$ so T(n) is $\Theta(n^{\log_b a}) = \Theta(n^{\log_2 8}) = \Theta(n^3)$

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Strassen's algorithm

- Strassen's algorithm
  - Multiply 2x2 matrices using 7 instead of 8 multiplications (and lots more than 4 additions)
  - $T(n)=7\ T(n/2)+cn^2$
    - $7>2^2$ so T(n) is $\Theta(n^{\log_2 7})$ which is $O(n^{2.81})$
  - Fastest algorithms theoretically use $O(n^{2.376})$ time
    - not practical but Strassen's is practical provided calculations are exact and we stop recursion when matrix has size about 100 (maybe 10)

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The algorithm

- $P_1=A_{12}(B_{11}+B_{21})$ $\quad$ $P_2=A_{21}(B_{12}+B_{22})$
- $P_3=(A_{11} - A_{12})B_{11}$ $\quad$ $P_4=(A_{22} - A_{21})B_{22}$
- $P_5=(A_{22} - A_{12})(B_{21} - B_{22})$
- $P_6=(A_{11} - A_{21})(B_{12} - B_{11})$
- $P_7= (A_{21} - A_{12})(B_{11}+B_{22})$
- $C_{11}=P_1+P_3$ $\quad$ $C_{12}=P_2+P_3+P_6 - P_7$
- $C_{21}= P_1+P_4+P_5+P_7$ $\quad$ $C_{22}=P_2+P_4$

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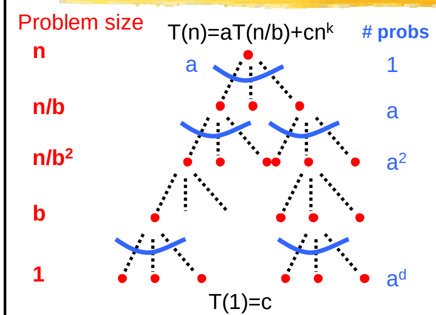
Proving Master recurrence



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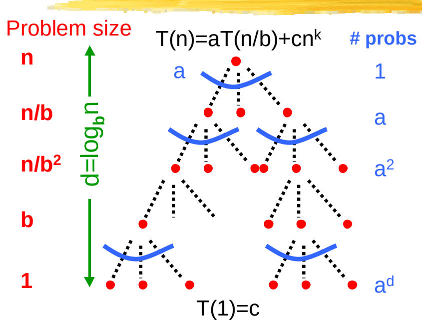
Proving Master recurrence



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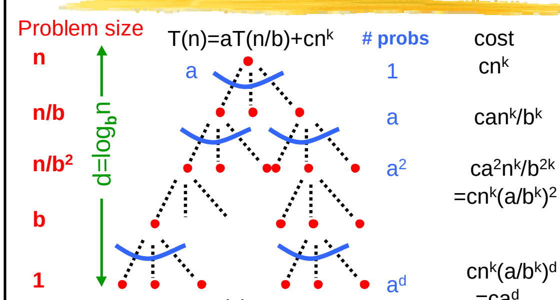
Proving Master recurrence



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Total Cost

- Geometric series
  - ratio a/b^k
  - $d+1=\log_b n+1$ terms
  - first term cn^k, last term ca^d
- If $a/b^k=1$, all terms are equal $T(n)$ is $\Theta(n^k \log n)$
- If $a/b^k<1$, first term is largest $T(n)$ is $\Theta(n^k)$
- If $a/b^k>1$, last term is largest
  $T(n)$ is $\Theta(a^d)=\Theta(a^{\log_b n})=\Theta(n^{\log_b a})$
  - (To see this take $\log_b$ of both sides)

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Quicksort

- Quicksort(X,left,right)

```
    if left < right
        split=Partition(X,left,right)
        Quicksort(X,left,split-1)
        Quicksort(X,split+1,right)
```

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Partition - two finger algorithm

- Partition(X,left,right)
  choose a random element to be a **pivot** and pull it out of the array, say at left end
  maintain two fingers starting at each end of the array
  slide them towards each other until you get a pair of elements where right finger has a smaller element and left finger has a bigger one (when compared to pivot)
  swap them and repeat until fingers meet
  put the pivot element where they meet

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Partition - two finger algorithm

- Partition(X,left,right)

```
    swap X[left],X[random(left,right)]
    pivot ← X[left];  L ← left; R ←right
    while L<R do
        while (X[L] ≤ pivot & L ≤ right) do
            L ← L+1
        while (X[R] > pivot & R ≥ left) do
            R ← R-1
        if L>R then swap X[L],X[R]
    swap X[left],X[R]
    return R
```

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In practice

- often choose pivot in fixed way as
  - middle element for small arrays
  - median of 1st, middle, and last for larger arrays
  - median of 3 medians of 3 (9 elements in all) for largest arrays
- four finger algorithm is better
  - also maintain two groups at each end of elements equal to the pivot
    - swap them all into middle at the end of Partition
  - equal elements are bad cases for two fingers

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Quicksort Analysis

- Partition does **n-1** comparisons on a list of length **n**
  - pivot is compared to each other element
- If **pivot** is **i**th largest then two subproblems are of size **i-1** and **n-i**
- Pivot is equally likely to be any one of **1**st through **n**th largest

$$T(n)=n-1+\frac{1}{n}\sum_{i=1}^{n}(T(i-1)+T(n-i))$$

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Quicksort analysis

$$T(n) = n-1+\frac{1}{n}\sum_{i=1}^{n}\left(T(i-1)+T(n-i)\right)$$

$$= n\text{-}1+\frac{2\,T(1)+2\,T(2)+\ldots+2\,T(n\text{-}1)}{n}$$

$$\therefore nT(n) = n(n\text{-}1)+2\,T(1)+2\,T(2)+\ldots+2\,T(n\text{-}1)$$

$$(n+1)T(n+1) = (n+1)n+2\,T(1)+2\,T(2)+\ldots+2\,T(n)$$

$$\therefore (n+1)T(n+1)\text{-}nT(n) = 2\,T(n)+2n$$

$$(n+1)T(n+1) = (n+2)T(n)+2n$$

$$\therefore \frac{T(n+1)}{n+2} = \frac{T(n)}{n+1}+\frac{2n}{(n+1)(n+2)}$$

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Quicksort analysis

Let $Q(n) = \frac{T(n)}{n+1}$

$$\therefore Q(n+1) \le Q(n)+\frac{2}{n+1}$$

$$\therefore Q(n) \le 2(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}) = 2H_n \approx 2\ln n = 1.38\log_2 n$$

(Recall that $\ln n = \int_1^n 1/x\ dx$)

$$\therefore T(n) \approx 1.38\,n\log_2 n$$

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