CSE 417: Algorithms and Computational Complexity

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Lecture 3

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Working with O-Ω-Θ notation

- Claim: For any a, b>1 $\log_a n$ is $\Theta(\log_b n)$
  - $\log_a n = \log_a b \log_b n$ so letting $c=\log_a b$ we get that $c\log_b n \le \log_a n \le c\log_b n$
- Claim: For any a and b>0, $(n+a)^b$ is $\Theta(n^b)$
  - $(n+a)^b \le (2n)^b$ for $n \ge |a|$
    $= 2^b n^b = cn^b$ for $c=2^b$ so $(n+a)^b$ is $O(n^b)$
  - $(n+a)b \ge (n/2)^b$ for $n \ge 2|a|$
    $=2^{-b}n^b = c'n$ for $c=2^{-b}$ so $(n+a)^b$ is $\Omega(n^b)$

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Solving recurrence relations

- e.g. $T(n)=T(n-1)+f(n)$ for $n \ge 1$
  $T(0)=0$
  - solution is $T(n)= \sum_{i=1}^{n} f(i)$
- Insertion sort: $T_n(i)=T_n(i-1)+i-1$
  - so $T_n(n)= \sum_{i=1}^{n}(i-1) = n(n-1)/2$

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Arithmetic Series

- S= 1 + 2 + 3 + ... + (n-1)
- S= (n-1)+(n-2)+(n-3)+ ... + 1
- 2S=n + n + n + ... + n {n-1 terms}
- 2S=n(n-1) so S=n(n-1)/2
- Works generally when f(i)=ai+b for all i
- Sum = average term size x # of terms

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Geometric Series

- $f(i)=a r^{i-1}$
- $S= a + ar + ar^2 + ... + ar^{n-1}$
- $rS= \quad ar + ar^2 + ... + ar^{n-1} + ar^n$
- $(r-1)S=ar^n - a$ so $S=a (r^n -1)/(r-1)$
- If r is a constant bounded away from 1
  - S is a constant times largest term in series

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Mixed recurrences

- $f(i)= i2^i$
- $S=1\ 2^1+2\ 2^2+... + n2^n$
- $2S= \quad 1\ 2^2+...+ (n-1)2^n + n2^{n+1}$
- $S= n2^{n+1}- (2 + 2^2 +...+ 2^n)$
  $=n2^{n+1}-(2^{n+1}-2)$
  $= (n-1)2^{n+1}+2$

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Guess & Verify

- $F_n=F_{n-1}+F_{n-2}$ for $n\geq 2$, $F_0=0$, $F_1=1$
- Guess that F_n is of the form α^n
  - therefore must have $\alpha^n=\alpha^{n-1}+\alpha^{n-2}$
  - i.e. $\alpha^2+\alpha+1=0$ $\quad \alpha=\frac{1\pm\sqrt{5}}{2}$
  - characteristic eqn
- $F_n=A(\frac{1+\sqrt{5}}{2})^n+B(\frac{1-\sqrt{5}}{2})^n$

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Guess & Verify

- $F_n=A(\frac{1+\sqrt{5}}{2})^n+B(\frac{1-\sqrt{5}}{2})^n$
- Solve: n=0: $A+B=0=F_0$
  n=1: $(A+B+(A-B)\sqrt{5})/2=1=F_1$
- Therefore $A=1/\sqrt{5}$ $\quad B=-1/\sqrt{5}$
- Now prove the whole thing by induction

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Repeated Substitution

- $T(n)=n+3T(n/4)$
- $= n+ 3(n/4+3T(n/16))$
- $= n+ 3n/4+ 9T(n/16)$
- $= n+ 3n/4+ 9n/16+27T(n/64)$
- Geometric series:
  - O(log n) terms
  - largest term n
  - $T(n)=\Theta(n)$

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Master Divide and Conquer Recurrence

- If $T(n)=aT(n/b)+cn^k$ for $n>b$ then
  - if $a>b^k$ then T(n) is $\Theta(n^{\log_b a})$
  - if $a<b^k$ then T(n) is $\Theta(n^k)$
  - if $a=b^k$ then T(n) is $\Theta(n^k \log n)$
- Works even if it is $\lceil n/b \rceil$ instead of n/b.

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Examples

- e.g. $T(n)=2T(n/2)+3n$ (Mergesort)
  - $2=2^1$ so $T(n)=\Theta(n\log n)$
- e.g. $T(n)=T(n/2) +1$ (Binary search)
  - $1=2^0$ so $T(n)=\Theta(\log n)$
- e.g. $T(n)=3T(n/4)+n$
  - $3<4^1$ so $T(n)=\Theta(n)$

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