CSE 417: Algorithms and Computational Complexity

Winter 2001
Lecture 24
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Steps to Proving Problem R is NP-complete

- Show R is NP-hard:
  - State:'Reduction is from NP-hard Problem L'
  - Show what the map is
  - Argue that the map is polynomial time
  - Argue correctness: two directions Yes for L implies Yes for R and vice versa.
- Show R is in NP
  - State what hint is and why it works
  - Argue that it is polynomial-time to check.

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Problems we already know are NP-complete

- Satisfiability
- Independent-Set
- Clique
- Vertex Cover

- There are 1000's of practical problems that are NP-complete, e.g. scheduling, optimal VLSI layout etc.

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A particularly useful problem for proving NP-completeness

- 3-SAT: Given a CNF formula F having precisely 3 variables per clause (i.e., in 3-CNF), is F satisfiable?

- Claim: 3-SAT is NP-complete
- Proof:
  - 3-SAT $\in$ NP
    - Hint is a satisfying assignment
    - Just like Satisfiability it is polynomial-time to check the hint

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Satisfiability $\leq^P$ 3-SAT

- Reduction:
  - mapping CNF formula F to another CNF formula G that has precisely 3 variables per clause.
    - G has one or more clauses for each clause of F
    - G will have extra variables that don't appear in F
      - for each clause C of F there will be a different set of variables that are used only in the clauses of G that correspond to C

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Satisfiability $\leq^P$ 3-SAT

- Goal:
    - An assignment A to the original variables makes clause C true in F iff
      - there is an assignment to the extra variables that together with the assignment A will make all new clauses corresponding to C true.
- Define the reduction clause-by-clause
  - We'll use variable names z_j to denote the extra variables related to a single clause C to simplify notation
    - in reality, two different original clauses will not share z_j

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Satisfiability $\le^P$ 3-SAT

- For each clause C in F:
  - If C has 3 variables:
    - Put C in G as is
  - If C has 2 variables, e.g. $C=(x_1 \lor \neg x_3)$
    - Use a new variable z and put two clauses in G
      $(x_1 \lor \neg x_3 \lor z) \land (x_1 \lor \neg x_3 \lor \neg z)$
    - If original C is true under assignment A then both new clauses will be true under A
    - If new clauses are both true under some assignment B then the value of z doesn't help in one of the two clauses so C must be true under B

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Satisfiability $\le^P$ 3-SAT

- If C has 1 variables: e.g. $C=x_1$
  - Use two new variables z_1, z_2 and put 4 new clauses in G
    $(x_1 \lor \neg z_1 \lor \neg z_2) \land (x_1 \lor \neg z_1 \lor z_2) \land (x_1 \lor z_1 \lor \neg z_2) \land (x_1 \lor z_1 \lor z_2)$
  - If original C is true under assignment A then all new clauses will be true under A
  - If new clauses are all true under some assignment B then the values of z_1 and z_2 doesn't help in one of the 4 clauses so C must be true under B

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Satisfiability $\le^P$ 3-SAT

- If C has $k \ge 4$ variables: e.g. $C=(x_1 \lor \ldots \lor x_k)$
  - Use k-3 new variables $z_2,\ldots,z_{k-2}$ and put k-2 new clauses in G
    $(x_1 \lor x_2 \lor z_2) \land (\neg z_2 \lor x_3 \lor z_3) \land (\neg z_3 \lor x_4 \lor z_4) \land \ldots \land (\neg z_{k-3} \lor x_{k-2} \lor z_{k-2}) \land (\neg z_{k-2} \lor x_{k-1} \lor x_k)$
  - If original C is true under assignment A then some x_i is true for $i \le k$. By setting z_j true for all $j<i$ and false for all $j \ge i$, we can extend A to make all new clauses true.
  - If new clauses are all true under some assignment B then some x_i must be true for $i \le k$ because $z_2 \land (\neg z_2 \lor z_3) \land \ldots \land (\neg z_{k-3} \lor z_{k-2}) \land \neg z_{k-2}$ is not satisfiable

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Graph Colorability

- Defn: Given a graph $G=(V,E)$, and an integer k, a k-coloring of G is
  - an assignment of up to k different colors to the vertices of G so that the endpoints of each edge have different colors.
- 3-Color: Given a graph $G=(V,E)$, does G have a 3-coloring?
- Claim: 3-Color is NP-complete
- Proof: 3-Color is in NP:
  - Hint is an assignment of red,green,blue to the vertices of G
  - Easy to check that each edge is colored correctly

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3-SAT $\le^P$ 3-Color

- Reduction:
  - We want to map a 3-CNF formula F to a graph G so that
    - G is 3-colorable iff F is satisfiable

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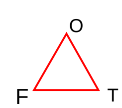
3-SAT $\le^P$ 3-Color



Base Triangle

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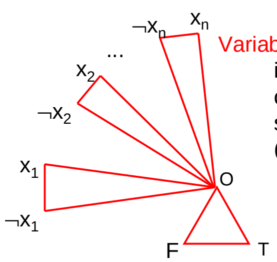
3-SAT $\leq^P$ 3-Color



Variable Part:
in 3-coloring, variable colors correspond to some truth assignment (same color as T or F)

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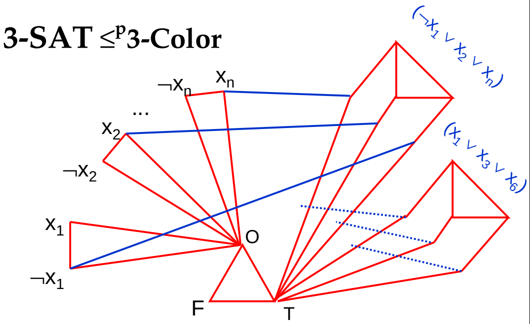
3-SAT $\leq^P$ 3-Color



Clause Part:
Add one 6 vertex gadget per clause connecting its ‘outer vertices’ to the literals in the clause

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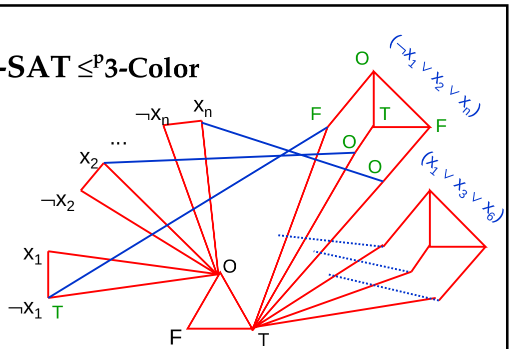
3-SAT $\leq^P$ 3-Color



Any truth assignment satisfying the formula can be extended to a 3-coloring of the graph

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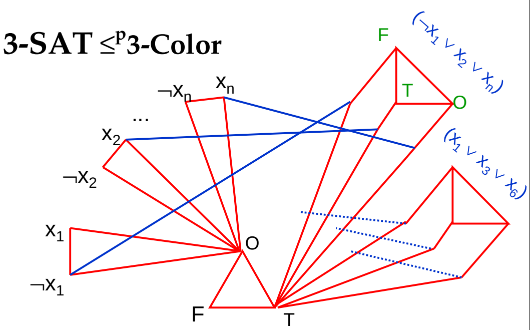
3-SAT $\leq^P$ 3-Color



Any 3-coloring of the graph colors each gadget triangle using each color

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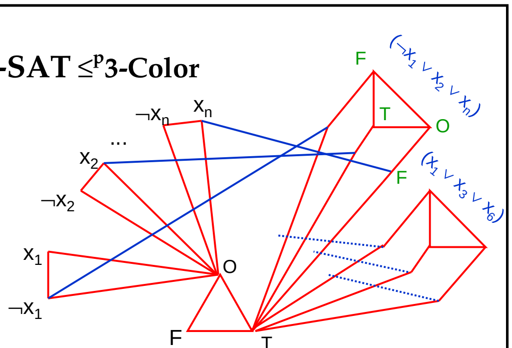
3-SAT $\leq^P$ 3-Color



Any 3-coloring of the graph has an F opposite the O color in the triangle of each gadget

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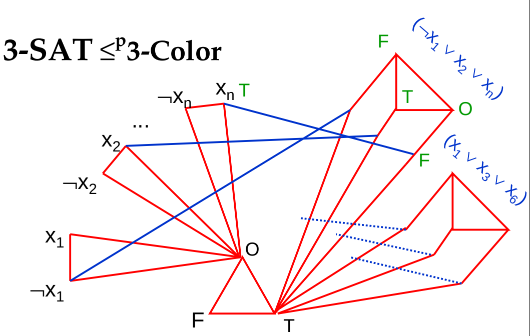
3-SAT $\leq^P$ 3-Color



Any 3-coloring of the graph has T at the other end of the blue edge connected to the F

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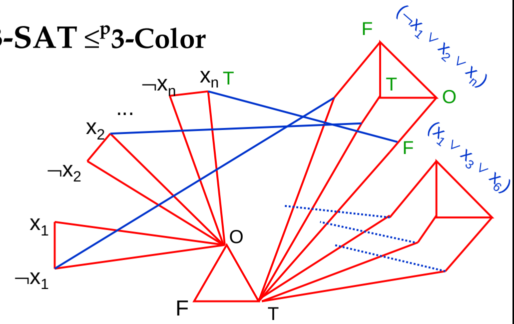
3-SAT $\leq^P$ 3-Color



Any 3-coloring of the graph yields a satisfying assignment to the formula

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