Number Formats

CSE 410, Spring 2006
Computer Systems

http://www.cs.washington.edu/education/courses/410/06sp/

Reading and References

- *Computer Organization and Design, Patterson and Hennessy*
  » Section 3.1 Introduction to Arithmetic for Computers
  » Section 3.2 Signed and unsigned numbers
  » Section 3.3 Addition and Subtraction
  » Section 3.6 Floating Point through page 197
  » Section 3.9 Concluding Remarks

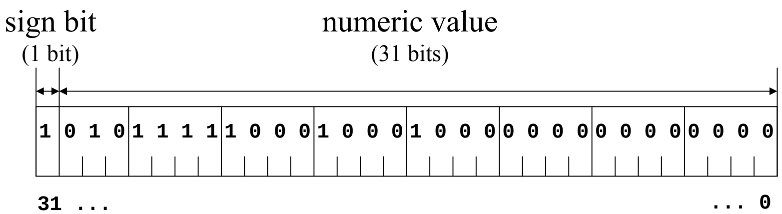
Signed Numbers

- We have already talked about unsigned binary numbers
  » each bit position represents a power of 2
  » range of values is 0 to 2^n-1
- How can we indicate negative values?
  » two states: positive or negative
  » a binary bit indicates one of two states: 0 or 1
  ⇒ use one bit for the sign bit

Where is the sign bit?

- Could use an additional bit to indicate sign
  » each value would require 33 bits
  » would really foul up the hardware design
- Could use any bit in the 32-bit word
  » any bit but the left-most (high order) would complicate the hardware tremendously
- ∴ The high order bit (left-most) is the sign bit
  » remaining bits indicate the value

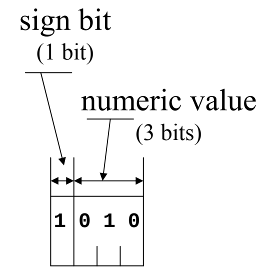
Format of 32-bit signed integer



- Bit 31 is the sign bit
  - » 0 for positive numbers, 1 for negative numbers
  - » aka most significant bit (msb), high order bit

Example: 4-bit signed numbers

Hex	Bin	Unsigned Decimal	Signed Decimal
F	1111	15	-1
E	1110	14	-2
D	1101	13	-3
C	1100	12	-4
B	1011	11	-5
A	1010	10	-6
9	1001	9	-7
8	1000	8	-8
7	0111	7	7
6	0110	6	6
5	0101	5	5
4	0100	4	4
3	0011	3	3
2	0010	2	2
1	0001	1	1
0	0000	0	0



Two's complement notation

- Note special arrangement of negative values
- One zero value, one extra negative value
- The representation is exactly what you get by doing a subtraction

```
Decimal      Binary

   1           0001
-  7      -    0111
----           ----
  -6           1010
```


Why "two's" complement?

- In an n-bit binary word, negative x is represented by the value of 2^n-x. The radix (or base) is 2.
  - » Wikipedia: "The **radix complement** of an *n* digit number *y* in radix *b* is $b^n - y$. Adding this to *x* results in the value $x + b^n - y$ or $x - y + b^n$. Assuming $y \leq x$, the result will always be greater than b^n and dropping the initial '1' is the same as subtracting b^n, making the result $x - y + b^n - b^n$ or just $x - y$, the desired result."
- 4-bit example

  $2^4 = 16$. What is the representation of -6?

```
Decimal      Binary

  16          10000
-  6      -    0110
----           ----
  10           1010
```

Negating a number

- Given x, how do we represent negative x?

```
        negative(x) = 2ⁿ-x
and x+complement(x) = 2ⁿ-1
so      negative(x) = 2ⁿ-x = complement(x)+1
```

- The easy shortcut
  - » write down the value in binary
  - » complement all the bits
  - » add 1

Example: the negation shortcut

```
 decimal 6 = 0110 = +6
complement = 1001
     add 1 = 1010 = -6

decimal -6 = 1010 = -6
complement = 0101
     add 1 = 0110 = +6
```


Signed and Unsigned Compares

Hex	Bin	Unsigned Decimal	Signed Decimal
F	1111	15	-1
E	1110	14	-2
D	1101	13	-3
C	1100	12	-4
B	1011	11	-5
A	1010	10	-6
9	1001	9	-7
8	1000	8	-8
7	0111	7	7
6	0110	6	6
5	0101	5	5
4	0100	4	4
3	0011	3	3
2	0010	2	2
1	0001	1	1
0	0000	0	0

```
add     $t0,$zero,-1
li      $t1,7
slt     $t2,$t0,$t1    # t2 = 1
sltu    $t3,$t0,$t1    # t3 = 0
```

Note: using 4-bit signed numbers in this example. The same relationships exist with 32-bit signed values.

Loading bytes

- Unsigned: **lbu $reg, a($reg)**
  - » the byte is 0-extended into the register

0000 0000	0000 0000	0000 0000	xxxx xxxx

- Signed: **lb $reg, a($reg)**
  - » bit 7 is extended through bit 31

0000 0000	0000 0000	0000 0000	0xxx xxxx

1111 1111	1111 1111	1111 1111	1xxx xxxx

Why Floating Point?

- The numbers we have talked about so far have all been integers in the range 0 to 4B or -2B to +2B
- What about numbers outside that range?
  - » population of the planet: 6 billion+
- What about numbers that have a fractional part in addition to the integer part?
  - » $\pi = 3.1415926535...$

Could use scaling to get fractions

- Assume that every numeric value in memory was scaled by a factor of 1000

  3000 => represents 3.000

  3010 => represents 3.010
- Problems
  - » one scale factor for all numbers?
  - » impossible to choose one "best" scale factor for all numbers that we might want to represent

A scale factor for each number

- This is the same as scientific notation
  - » 6×10^9, 3.1415926535×10^0
- A floating point number contains two parts
  - » mantissa (or significand): the value
  - » exponent: the exponent of the scale factor
- Normalized form
  - » a non-zero single digit before the decimal point

"Binary scientific notation"

- The computer only stores binary numbers
  - » So we use powers of 2 rather than 10
  - » Normalized numbers have a leading 1
- $6{,}000{,}000{,}000 = 6.0 \times 10^9$
  - » $1.3969838619_{10} \times 2^{32}$
- $\pi \cong 3.14159265358979323846264338 3$
  - » $1.57079632679489661923132169163975 \times 2^1$

Storage format: fixed width fields

- How big can the exponent be?
  - » what is the range it represents?
- How big can the mantissa be?
  - » what are the values it represents?
- We have to select a storage format and allocate specific fields to various purposes
  - » single precision: one 32-bit word
  - » double precision: two 32-bit words

IEEE 754 Standard

- Chaos in the 70s and 80s as each system designer chose new formats and rules
- IEEE 754 standard
  - » format of the fields
  - » rounding: up, down, towards 0, nearest
  - » exceptional values: ±infinity, NaN (not a number)
  - » action to take on exceptional values

Floating Point Storage

- Single Precision
  - » one word (32 bits)
- Double Precision
  - » two words (64 bits)
  - » the order of the words depends on endianness of the machine being used
- Defined by IEEE 754

Single Precision Format

s	exponent	mantissa
1	*8 bits*	*23 bits*
0	1 0 0 0 0 0 1 1	1 1 1 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

Double Precision Format

s	exponent	mantissa (high order bits)
1	*11 bits*	*20 bits*
0	1 0 0 0 0 0 0 1 1 1 1 1	0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

mantissa (low order bits)
32 bits
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

Double Precision Mantissa Fields

- Sign bit
  - » 1 bit sign for the value
- Mantissa
  - » 52 bits for the value
  - » by definition, the leading digit is always a 1
  - » so we don't need to actually store it
  - » and we actually have 53 bits of information

Double Precision Exponent Field

- Field range
  - » 11 bits: range $2^{11} = 2048$ possible values
- Special values
  - » exponent = 2047 $\Rightarrow$ value=special (inf, NaN)
  - » exponent = 0 $\Rightarrow$ value=0

Biased Notation

- Need exponent range - negative and positive
- If positive exponents are bigger numbers than the negative exponents, then floating point numbers can be sorted as integers
- Exponent is stored as (E+1023)
  - » most positive exponent is +1023 (stored as 2046)
  - » most negative exponent is -1022 (stored as 1)
  - » this is not two's complement notation

Example: 6,174,015,488

- 6174015488
  = $6.174015488 \times 10^9 = 1.4375_{10} \times 2^{32}$
- Exponent
  = $32+1023 = 1055 = 41F_{16}$
- Mantissa
  = $.4375_{10} = .0111_2 = 7_{16}$

6,174,015,488

s	exponent			mantissa (high order bits)				
1	*11 bits*			*20 bits*				
0	1 0 0	0 0 0 1	1 1 1 1	0 1 1 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
hex: 4		1	F	7	0	0	0	0

mantissa (low order bits)							
32 bits							
0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0

Roundoff Error

- Adding a very small floating point number to a very large floating point number may not have any effect
  - » any one number has only 53 significant bits
- Adding a number with a fractional part to another number over and over will probably never yield an exactly integer result
  - » so don't use floating point loop indexes

Loss of precision

<u>1101 0000 0000 0000.</u>0000 0000 0000 0000 = $1.101_2 \times 2^{15}$

0000 0000 0000 0000<u>.0000 0000 0000 1101</u> = $1.101_2 \times 2^{-13}$

- These are not unusual numbers
  53248 and 0.0001983642578125
- Very few bits of mantissa required
- But their sum requires a mantissa with at least <u>32 bits</u> or there will lost significant bits