CSE 390Z: Mathematics for Computation Workshop

Week 3 Workshop Problems Solutions

Conceptual Review

(a) Inference Rules:

Modus Ponens:	$\frac{A\ ;\ A\to B}{\therefore\ B}$	Direct Proof:	$\frac{A\Rightarrow B}{\therefore\ A\to B}$
Eliminate $\wedge$:	$\frac{A\wedge B}{\therefore\ A,\ B}$	Introduce $\wedge$:	$\frac{A\ ;\ B}{\therefore\ A\wedge B}$
Proof by Cases:	$\frac{A\vee B\ ;\ A\to C\ ;\ B\to C}{\therefore\ C}$	Introduce $\vee$:	$\frac{A}{\therefore\ A\vee B,\ B\vee A}$
Eliminate $\vee$:	$\frac{A\vee B\ ;\ \neg A}{\therefore\ B}$		
Principium Contradictionis	$\frac{\neg A\ ;\ A}{\therefore\ F}$	Reductio Ad Absurdum	$\frac{B\Rightarrow F}{\therefore\ \neg B}$
Ex Falso Quodlibet	$\frac{F}{\therefore\ A}$	Ad Litteram Verum	$\frac{}{\therefore\ T}$
Tautology	$\frac{A\equiv T}{\therefore\ A}$	Equivalent	$\frac{A\equiv B\ ;\ B}{\therefore\ A}$
Intro $\exists$:	$\frac{P(c) \text{ for some } c}{\therefore\ \exists x P(x)}$	Eliminate $\forall$:	$\frac{\forall x P(x)}{\therefore\ P(a) \text{ for any } a}$
Eliminate $\exists$*:	$\frac{\exists x P(x)}{\therefore\ P(c) \text{ for a new } c}$	Intro $\forall$*:	$\frac{P(a);\ a \text{ is arbitrary}}{\therefore\ \forall x P(x)}$

* You haven't seen these rules in lecture yet.

(b) Given $A \wedge B$, prove $A \vee B$

Solution:

1. $A \wedge B$ (Given)
2. A (Elim $\wedge$: 1.)
3. $A \vee B$ (Intro $\vee$: 2.)

(c) What is the purpose of the direct proof rule? How do you use it? Why are we allowed to do this?

Solution:

We use the direct proof rule to prove an implication (e.g., $p \to q$).

First, we assume the premise of the implication (e.g., p) is true. We indent this part of our solution to provide a visual indication that we're working under an assumption. Now that we've assumed the premise holds, we can use it the same way we would use any other expression in our proof. We then apply our inference rules and arrive at our conclusion (e.g., q). We step back outside our cozy indented assumption box and conclude that the implication $p \to q$ holds.

Now, why can we do this? When we indent our proof, we don't know if our premise p is true or not. We're not asserting that it is true. We're asking the reader to pretend for a second that p is true.

When we use this assumption to show that q is true, do we know that q is actually true? NO! That conclusion is based on the assumption that p is true, and we made that up!! What we do know is that *if* p is true *then* q must be true, so we can conclude that $p \to q$.

(d) Given $P \to R$, $R \to S$, prove $P \to S$.

Solution:

1. $P \to R$ (Given)
2. $R \to S$ (Given)
   - 3.1 P (Assumption)
   - 3.2 R (Modus Ponens: 3.1, 1)
   - 3.3 S (Modus Ponens: 3.2, 2)
3. $P \to S$ (Direct Proof Rule)

(e) What is a common way to use Reductio Ad Absurdum and Principium Contradictionis together to prove that a proposition is false?

Solution:

Reductio Ad Absurdum: If we assume B is true and prove false, B cannot be true.
Principium Contradictionis: If we prove that A is both false and true, we know something went wrong, so we conclude false.
We csn assume that B is true, and just like with the Direct Proof Rule, we step into indented assumption land where we now take that assumption as fact. Then, in our indented assumption land, we prove that A and $\neg A$ are both true for some proposition A, which makes zero sense. (Note that we may have already shown one of A or $\neg A$ outside of indented assumption land). This allows us to conclude false using Principium Contradictionis, still inside indented assumption land. Since we assumed B was true and proved false, we can leave our indented assumption land and use Reductio Ad Absurdum to conclude B must not be true (in other words, we conclude $\neg B$).

1. Formal Proofs: Modus Ponens

(a) Prove that given $p \to q$, $\neg s \to \neg q$, and p, we can conclude s.
Hint: You may need to use a contrapositive at some point.

Solution:

1. $p \to q$ (Given)
2. $\neg s \to \neg q$ (Given)
3. p (Given)
4. q (Modus Ponens; 3,1)
5. $q \to s$ (Contrapositive; 2)
6. s (Modus Ponens; 4,5)

(b) Prove that given $\neg\ s \to (q \vee p)$, $\neg p$, and $\neg s$, we can conclude q.

Solution:

1. $\neg\ s \rightarrow (q \vee p)$ (Given)
2. $\neg p$ (Given)
3. $\neg s$ (Given)
4. $q \vee p$ (Modus Ponens; 3,1)
5. $p \vee q$ (Commutativity; 4)
6. q (Elim $\vee$; 5,2)

2. Formal Proofs: Direct Proof Rule

(a) Prove that given $p \rightarrow q$, we can conclude $(p \wedge r) \rightarrow q$

Solution:

1. $p \rightarrow q$ (Given)
   2.1 $p \wedge r$ (Assumption)
   2.2 p (Elim $\wedge$; 2.1)
   2.3 q (Modus Ponens; 2.2, 1)
2. $(p \wedge r) \rightarrow q$ (Direct proof rule)

(b) Prove that given $p \vee q$, $q \rightarrow r$, and $r \rightarrow s$, we can conclude $\neg p \rightarrow s$.

Solution:

1. $p \vee q$ (Given)
2. $q \rightarrow r$ (Given)
3. $r \rightarrow s$ (Given)
   4.1 $\neg p$ (Assumption)
   4.2 q (Elim $\vee$; 1, 4.1)
   4.3 r (Modus Ponens; 4.2, 2)
   4.4 s (Modus Ponens; 4.3, 3)
4. $\neg p \rightarrow s$ (Direct proof rule)

(c) Prove that $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \wedge q) \rightarrow r)$
You can **not** use any logical equivalences in your solution.

Solution:

   1.1 p →(q →r) (Assumption)
      1.2.1 $p \wedge q$ (Assumption)
      1.2.2 p (Elim $\wedge$; 1.2.1)
      1.2.3 q (Elim $\wedge$; 1.2.1)
      1.2.4 $q \rightarrow r$ (Modus Ponens; 1.2.2, 1.1)
      1.2.5 r (Modus Ponens; 1.2.3, 1.2.4)

1.2 $(p \wedge q) \rightarrow r$ (Direct Proof Rule)

1. $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \wedge q) \rightarrow r)$ (Direct Proof Rule)

3. Formal Proofs: Quantifiers

(a) Prove that $\forall x P(x) \rightarrow \exists x P(x)$. You may assume that the domain is nonempty.

Solution:

1.1. $\forall x P(x)$ (Assumption)

1.2. $P(a)$ (Elim $\forall$: 1.1)

1.3. $\exists x P(x)$ (Intro $\exists$: 1.2)

1. $\forall x P(x) \rightarrow \exists x P(x)$ (Direct Proof Rule)

(b) Given $\forall x(T(x) \rightarrow M(x))$ and $\forall x T(x)$, prove that $\exists x M(x)$.

Solution:

1. $\forall x(T(x) \rightarrow M(x))$ (Given)
2. $\forall x T(x)$ (Given)
3. $T(r)$* (Elim $\forall$; 2)
4. $T(r) \rightarrow M(r)$** (Elim $\forall$; 1)
5. $M(r)$ (Modus Ponens; 3, 4)
6. $\exists x M(x)$ (Intro $\exists$; 5)

* We can pick any value we want. We don't need anything special, so we pick a random thing in the domain and call it r.
** We can pick any value we want, so we pick the r from step 3.

(c) Given $\forall x(P(x) \rightarrow Q(x))$, prove that $(\forall x P(x)) \rightarrow (\exists y Q(y))$.

Solution:

1. $\forall x(P(x) \rightarrow Q(x))$ (Given)

2.1. $\forall x P(x)$ (Assumption)

2.2. $P(r)$* (Elim $\forall$; 2.1)

2.3. $P(r) \rightarrow Q(r)$** (Elim $\forall$; 1)

2.4. $Q(r)$ (Modus Ponens; 2.2, 2.3)

2.5. $\exists y Q(y)$ (Intro $\exists$; 2.4)

2. $(\forall x P(x)) \rightarrow (\exists y Q(y))$ (Direct Proof Rule)

* We can pick any value we want. We don't need anything special, so we pick a random thing in the domain and call it r.
** We can pick any value we want, so we pick the r from step 2.2

4. Formal Proofs: Latin Rules

(a) Show that $\neg(A \wedge B)$ follows from $\neg A \vee \neg B$
You can **not** use any logical equivalences in your solution.

Solution:

1. $\neg A \vee \neg B$ (Given)
    2.1 $A \wedge B$ (Assumption)
    2.2 A (Elim $\wedge$ 2.1)
    2.3 B (Elim $\wedge$ 2.1)
        2.4.1 $\neg A$ (Assumption)
        2.4.2 F (Contradiction 2.4.1, 2.2)
    2.4 $\neg A \rightarrow$ F (Direct Proof Rule)
        2.5.1 $\neg B$ (Assumption)
        2.5.2 F (Contradiction 2.5.1, 2.3)
    2.5 $\neg B \rightarrow$ F (Direct Proof Rule)
    2.6 F (Cases 1, 2.4, 2.5)
2. $\neg(A \wedge B)$ (Absurdum)

(b) Given $P \rightarrow Q$ and $\neg R$ prove that $P \rightarrow \neg(Q \rightarrow R)$
You can **not** use any logical equivalences in your solution.

Solution:

1. $P \rightarrow Q$ (Given)
2. $\neg R$ (Given)
    3.1 P (Assumption)
    3.2 Q (Modus Ponens 3.1, 1)
        3.3.1 $Q \rightarrow R$ (Assumption)
        3.3.2 R (Modus Ponens 3.2, 3.3.1)
        3.3.3 F (Contradiction 2, 3.3.2)
    3.3 $\neg(Q \rightarrow R)$ (Absurdum)
3. $P \rightarrow \neg(Q \rightarrow R)$ (Direct Proof Rule)

(c) Given $\neg C$, $D \rightarrow (E \vee C)$, $\neg C \rightarrow (A \wedge B)$, prove $\neg((D \wedge \neg E) \vee \neg A)$
You can **not** use any logical equivalences in your solution.

Solution:

1. $\neg C$ (Given)
2. $D \rightarrow (E \vee C)$ (Given)
3. $\neg C \rightarrow (A \wedge B)$ (Given)
4. $A \wedge B$ (Modus Ponens 1, 3)
    5.1 $(D \wedge \neg E) \vee \neg A$ (Assumption)
        5.2.1 $D \wedge \neg E$ (Assumption)

5.2.2 D (Elim $\wedge$ 5.2.1)
5.2.3 $\neg E$ (Elim $\wedge$ 5.2.1)
5.2.4 $E \vee C$ (Modus Ponens 5.2.2, 2)
5.2.5 C (Elim $\vee$ 5.2.4, 5.2.3)
5.2.6 F (Contradiction 1, 5.2.5)
5.2 $(D \wedge \neg E) \to$ F (Direct Proof Rule)
5.3.1 $\neg A$ (Assumption)
5.3.2 A (Elim $\wedge$ 4)
5.3.3 F (Contradiction 5.3.1, 5.3.2)
5.3 $\neg A \to$ F (Direct Proof Rule)
5.4 F (Cases 5.1, 5.2, 5.3)
5. $\neg((D \wedge \neg E) \vee \neg A)$ (Absurdum)

5. Formal Proofs: Challenge

Given $\forall x\ (P(x) \vee Q(x))$ and $\forall y\ (\neg Q(y) \vee R(y))$, prove $\exists x\ (P(x) \vee R(x))$. You may assume that the domain is not empty.

Hint: You can cite logical equivalences too.

Solution:

1. $\forall x\ (P(x) \vee Q(x))$ [Given]
2. $\forall y\ (\neg Q(y) \vee R(y))$ [Given]
3. $P(a) \vee Q(a)*$ [Elim $\forall$: 1]
4. $\neg Q(a) \vee R(a) **$ [Elim $\forall$: 2]
5. $Q(a) \to R(a)$ [Law of Implication: 4]
6. $\neg\neg P(a) \vee Q(a)$ [Double Negation: 3]
7. $\neg P(a) \to Q(a)$ [Law of Implication: 6]
    8.1. $\neg P(a)$ [Assumption]
    8.2. $Q(a)$ [Modus Ponens: 8.1, 7]
    8.3. $R(a)$ [Modus Ponens: 8.2, 5]
8. $\neg P(a) \to R(a)$ [Direct Proof Rule]
9. $\neg\neg P(a) \vee R(a)$ [Law of Implication: 8]
10. $P(a) \vee R(a)$ [Double Negation: 9]
11. $\exists x\ (P(x) \vee R(x))$ [Intro $\exists$: 10]

* We can pick any value we want. We don't need anything special, so we pick a random thing in the domain and call it a.

** We can pick any value we want. We intentionally choose the a from step 3.

6 Formal Proofs: More Quantifiers - Try this later

Note: These are very similar to the proofs you saw earlier, but require either the Intro $\forall$ or Elim $\exists$ rules.

(a) Given $\forall x(T(x) \to M(x))$ and $\exists x T(x)$, prove that $\exists x M(x)$.

Solution:

1. $\forall x(T(x) \to M(x))$ (Given)
2. $\exists x T(x)$ (Given)
3. $T(r)$* (Elim $\exists$; 2)
4. $T(r) \to M(r)$** (Elim $\forall$; 1)
5. $M(r)$ (Modus Ponens; 3, 4)
6. $\exists x M(x)$ (Intro $\exists$; 5)

* r is the value that satisfies $T(x)$
** We can pick any value we want. We intentionally pick the r from step 3.

(b) Given $\forall x(P(x) \to Q(x))$, prove that $(\exists x P(x)) \to (\exists y Q(y))$.

Solution:

1. $\forall x(P(x) \to Q(x))$ (Given)
   - 2.1. $\exists x P(x)$ (Assumption)
   - 2.2. $P(r)$* (Elim $\exists$; 2.1)
   - 2.3. $P(r) \to Q(r)$** (Elim $\forall$; 1)
   - 2.4. $Q(r)$ (Modus Ponens; 2.2, 2.3)
   - 2.5. $\exists y Q(y)$ (Intro $\exists$; 2.4)
2. $(\exists x P(x)) \to (\exists y Q(y))$ (Direct Proof Rule)

* r is the value that satisfies $P(x)$
** We can pick any value we want. We intentionally pick the r from step 2.2