CSE 390Z: Mathematics for Computation Workshop

Practice 311 Final Solutions

Name: ______________________________

UW ID: ______________________________

Instructions:

- This is a **simulated practice final**. You will **not** be graded on your performance on this exam.
- Nevertheless, please treat this as if it is a real exam. That means that you may not discuss with your neighbors, reference outside material, or use your devices during the next 100 minute period.
- There are 10 "graded" problems on this exam and 1 bonus problem.

This page is intentionally left blank.

1. Predicate Translation

Let the domain of discourse be numbers and mathematicians (math teachers and math students). Let the following predicates be defined:

Number(x) := x is a number
Prime(x) := x is a prime number
MathTeacher(x) := x is a math teacher
MathStudent(x) := x is a math student
Likes(x, y) := x likes y
You can also use $x = y$ and $x \neq y$.

For parts (a) and (b), translate the predicate logic statement into natural English.

(a) $\forall x \forall y((\text{Prime}(x) \land \text{Likes}(y, x)) \rightarrow \text{MathStudent}(y))$

Solution:

Only math students like prime numbers.

(b) $\exists x(\text{Prime}(x) \land \forall y((\text{MathTeacher}(y) \lor \text{MathStudent}(y)) \rightarrow \text{Likes}(y, x)))$

Solution:

There is a prime number that all mathematicians like.

For parts (c) and (d), translate the English sentence into predicate logic.

(c) There are at least two (different) prime numbers

Solution:

$\exists x \exists y(\text{Prime}(x) \land \text{Prime}(y) \land x \neq y)$

(d) Math teachers and math students don't like the same numbers.

Solution:

$\forall x \forall y \forall z((\text{MathTeacher}(x) \land \text{MathStudent}(y) \land \text{Number}(z)) \rightarrow \neg(\text{Likes}(x, z) \land \text{Likes}(y, z))))$
OR
$\neg \exists x \exists y \exists z(\text{Number}(x) \land \text{MathTeacher}(y) \land \text{MathStudent}(z) \land \text{Likes}(y, x) \land \text{Likes}(z, x))$

2. Formal Proof

Write a **formal proof** of the following statement:

For all integers a, b, c, if $a^2 \mid b$ and $b^3 \mid c$, then $a^6 \mid c$.

Solution:

Let x, y, z be arbitrary integers.

1.1.1	$(x^2 \mid y) \wedge (y^3 \mid z)$	Assumption
1.1.2	$x^2 \mid y$	Elim $\wedge$: 1.1.1
1.1.3	$y^3 \mid z$	Elim $\wedge$: 1.1.2
1.1.4	$\exists k(y = x^2 k)$	Def of divides: 1.1.2
1.1.5	$\exists k(z = y^3 k)$	Def of divides: 1.1.3
1.1.6	$y = x^2 s$	Elim $\exists$: 1.1.4
1.1.7	$z = y^3 t$	Elim $\exists$: 1.1.5
1.1.8	$z = (x^2 s)^3 t = x^6 (s^3 t)$	
1.1.9	$\exists k(z = x^6 k)$	Intro $\exists$: 1.1.8
1.1.10	$x^6 \mid z$	Undef of divides: 1.1.9
1.1	$((x^2 \mid y) \wedge (y^3 \mid z)) \rightarrow x^6 \mid z$	Direct Proof
1.	$\forall a \forall b \forall c(((a^2 \mid b) \wedge (b^3 \mid c)) \rightarrow a^6 \mid c)$	Intro $\forall$

3. English Proof

Write an **English proof** of the following statement:

For all integers n, $n^2 \equiv_4 0$ or $n^2 \equiv_4 1$

Solution:

Let n be an arbitrary integer. We proceed by cases.

Case 1: Suppose n is even.
By definition of even, $n = 2k$ for some integer k. Then $n^2 = (2k)^2 = 4k^2$. By definition of divides, $4 \mid n^2 - 0$, and by definition of congruence, $n^2 \equiv_4 0$.

Case 2: Suppose n is odd.
By definition of odd, $n = 2k + 1$ for some integer k. Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$. So $n^2 - 1 = 4(k^2 + k)$. By definition of divides, $4 \mid n^2 - 1$, and by definition of congruence, $n^2 \equiv_4 1$.

Since these cases are exhaustive, we have shown that $n^2 \equiv_4 0$ or $n^2 \equiv_4 1$. Since n was arbitrary, the claim holds for all integers n.

4. Induction I

Prove by induction that $(1+\pi)^n > 1 + n\pi$ for all integers $n \geq 2$.

Solution:

1. Let P(n) be the statement "$(1+\pi)^n > 1 + n\pi$". We prove P(n) holds for all integers $n \geq 2$ by induction.

2. Base Case ($n = 2$)

LHS: $(1+\pi)^2 = 1 + 2\pi + \pi^2$

RHS: $1 + 2\pi$

Since $\pi^2 > 0$ we have, $(1+\pi)^2 = 1 + 2\pi + \pi^2 > 1 + 2\pi$, so the base case holds.

3. Inductive Hypothesis: Suppose that P(k) holds for some arbitrary integer $k \geq 2$. Then $(1+\pi)^k > 1 + k\pi$.

4. Inductive Step:

$$\boxed{\text{Goal: Show } P(k+1)\text{, i.e. show } (1+\pi)^{k+1} > 1 + (k+1)\pi}$$

$$\begin{aligned}
(1+\pi)^{k+1} &= (1+\pi)(1+\pi)^k && \text{Definition of Exponent} \\
&> (1+\pi)(1+k\pi) && \text{By IH} \\
&= 1 + \pi + k\pi + k\pi^2 && \text{Algebra} \\
&= 1 + (k+1)\pi + k\pi^2 && \text{Algebra} \\
&> 1 + (k+1)\pi && \text{Since } k\pi^2 > 0
\end{aligned}$$

Thus $(1+\pi)^{k+1} > 1 + (k+1)\pi$ and P($k+1$) holds.

5. P(n) holds for all integers $n \geq 2$ by induction.

5. Set Theory

(a) Let A and B be sets. Write an **English proof** of the following claim using the Meta Theorem:

$$A \setminus \overline{B} = (A \cup \varnothing) \cap B$$

Hint: The empty set, $\varnothing$, is the set that contains no elements. In other words, for any x, $x \in \varnothing \equiv F$.

Solution:

The claim is equivalent to $\forall x(x \in (A \setminus \overline{B}) \leftrightarrow x \in ((A \cup \varnothing) \cap B))$.
Let x be arbitrary.

$$\begin{aligned}
x \in (A \setminus \overline{B}) &\equiv x \in A \wedge \neg(x \in \overline{B}) && \text{Def of Set Difference} \\
&\equiv x \in A \wedge \neg\neg(x \in B) && \text{Def of Complement} \\
&\equiv x \in A \wedge x \in B && \text{Double Negation} \\
&\equiv (x \in A \vee F) \wedge x \in B && \text{Identity} \\
&\equiv (x \in A \vee x \in \varnothing) \wedge x \in B && \text{Def of } \varnothing \\
&\equiv x \in (A \cup \varnothing) \wedge x \in B && \text{Def of Union} \\
&\equiv x \in (A \cup \varnothing) \cap B && \text{Def of Intersection}
\end{aligned}$$

Since x was arbitrary, we have shown that $A \setminus \overline{B} = (A \cup \varnothing) \cap B$.

Determine if the following claims are true or false. Explain your reasoning in 1-3 sentences.
You may include images or examples in your explanation. **You do not need to give a proof or disproof.**

(b) For all sets A, B: $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$.

Solution:

True. Both sets represent the set of elements that are in A or B but not in both. That is, both sets are equal to the set $\{x : x \in A \oplus x \in B\}$.

(c) For all sets A, B: $\mathcal{P}(A) \times \mathcal{P}(B) \subseteq \mathcal{P}(A \times B)$.

Solution:

False. Conceptually, the elements of $\mathcal{P}(A) \times \mathcal{P}(B)$ are ordered pairs of sets, while the elements of $\mathcal{P}(A \times B)$ are sets of ordered pairs. We can look at a counterexample too. Consider $A = \{1\}$ and $B = \{2\}$. $\mathcal{P}(A) = \{\varnothing, \{1\}\}$ and $\mathcal{P}(B) = \{\varnothing, \{2\}\}$, so $\mathcal{P}(A) \times \mathcal{P}(B) = \{(\varnothing, \varnothing), (\varnothing, \{2\}), (\{1\}, \varnothing), (\{1\}, \{2\})\}$ $A \times B = \{(1, 2)\}$, so $\mathcal{P}(A \times B) = \{\varnothing, \{(1, 2)\}\}$. Not only are these not the same set, they don't have any elements in common.

6. Induction II

Let the set S be recursively defined as follows:
Basis: $(0,0) \in S$
Recursive Step: If $(x,y) \in S$, then $(x+2, y+4) \in S$ and $(x+4, y+8) \in S$.

Prove that for all $(x,y) \in S$, $x+y$ is divisible by 3.

Solution:

Define $P((x,y))$ to be the claim $3 \mid (x+y)$. We will prove that $P((x,y))$ holds for all $(x,y) \in S$ by structural induction.

Base Case: $(x,y) = (0,0)$
$0+0 = 0 = 3*0$. So, $3|(x+y)$ and $P((0,0))$ holds.

Inductive Hypothesis: Suppose that $P((a,b))$ holds for some arbitrary $(a,b) \in S$. In other words, suppose $3|(a+b)$. By definition divides, this means $a+b = 3k$ for some $k \in \mathbb{Z}$

Inductive Step:

Goal: Show $P((a{+}2, b{+}4))$ and $P((a{+}4, b{+}8))$

P((a+2, b+4)):

$$\begin{aligned}(a+2)+(b+4) &= (a+b)+6 \\ &= 3k+6 && \text{By IH} \\ &= 3(k+2)\end{aligned}$$

Thus, by definition of divides, $3|((a+2)+(b+4))$ and $P((a+2, b+4))$ holds.

P((a+4, b+8)):

$$\begin{aligned}(a+4)+(b+8) &= (a+b)+12 \\ &= 3k+12 && \text{By IH} \\ &= 3(k+4)\end{aligned}$$

Thus, by definition of divides, $3|((a+4)+(b+8))$ and $P((a+4, b+8))$ holds.

Conclusion: Therefore, $P((x,y))$ holds for all $(x,y) \in S$.

7. Relations

For parts (a) and (b), consider the relation $R \subseteq \mathbb{Z} \times \mathbb{Z}$ defined by $(x, y) \in R$ iff $4|(x + y)$

(a) List 3 elements of R.

Solution:

- (0,0)
- (4,0)
- (2,2)

(b) List the properties that R has out of the following: reflexive, transitive, symmetric, antisymmetric. If R has a property, simply say so without further explanation. If R does not have a property, provide a counterexample without explaining further.

Solution:

Reflexive: No. (1,1) is not in R.
Transitive: No. (3,1) and (1,3) are in R, but (3,3) is not.
Symmetric: Yes.
Antisymmetric: No. (3,1) and (1,3) are both in R.

(c) Let R and S be symmetric relations on a set A. Write an **English proof** that $R \setminus S$ is symmetric.

Solution:

Let $(a, b) \in R \setminus S$ be arbitrary. By definition of set difference, $(a, b) \in R$ and $(a, b) \notin S$.

Since R is symmetric, $(b, a) \in R$.

Since S is symmetric, $(x, y) \in S \to (y, x) \in S$. Taking the contrapositive of this implication, we get $(y, x) \notin S \to (x, y) \notin S$. So, since $(a, b) \notin S$, $(b, a) \notin S$.

Since $(b, a) \in R$ and $(b, a) \notin S$, by definition of set difference, $(b, a) \in R \setminus S$.

Since (a, b) was arbitrary, we have shown that $R \setminus S$ is symmetric.

8. Languages

For parts (a), (b), (c) consider the language L containing all binary strings x with the following property:
If there is a 0 at position i in x, then there is a 1 at position $i + 2$ in x.
In other words, every time we see a 0, the character after the 0 can be anything, but the character after that has to be a 1.
Some strings in L are: ε, 011, 0011, 011011, 111011. Some strings not in L are: 0, 01, 001, 01011.
Notice that 001 is not in L because even though there is a 1 two characters after the first 0, there is not a 1 two characters after the second 0.

(a) Construct a regular expression that matches the strings in L.

Solution:

$(1 \cup 011 \cup 0011)^*$

There may be other valid solutions.

(b) Construct a CFG that generates L.

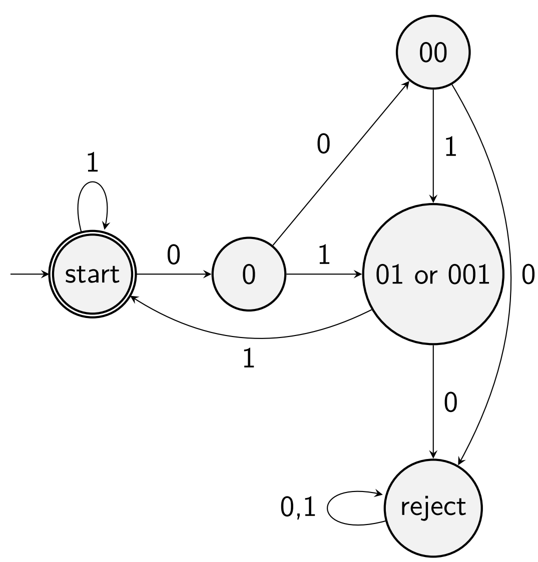
Solution:

Solution 1: **S** → **SS** | 1 | 011 | 0011 | ε
Solution 2: **S**→ 1**S** | 011**S** | 0011**S** | ε
There may be other valid solutions.

(c) Construct a DFA that recognizes L.

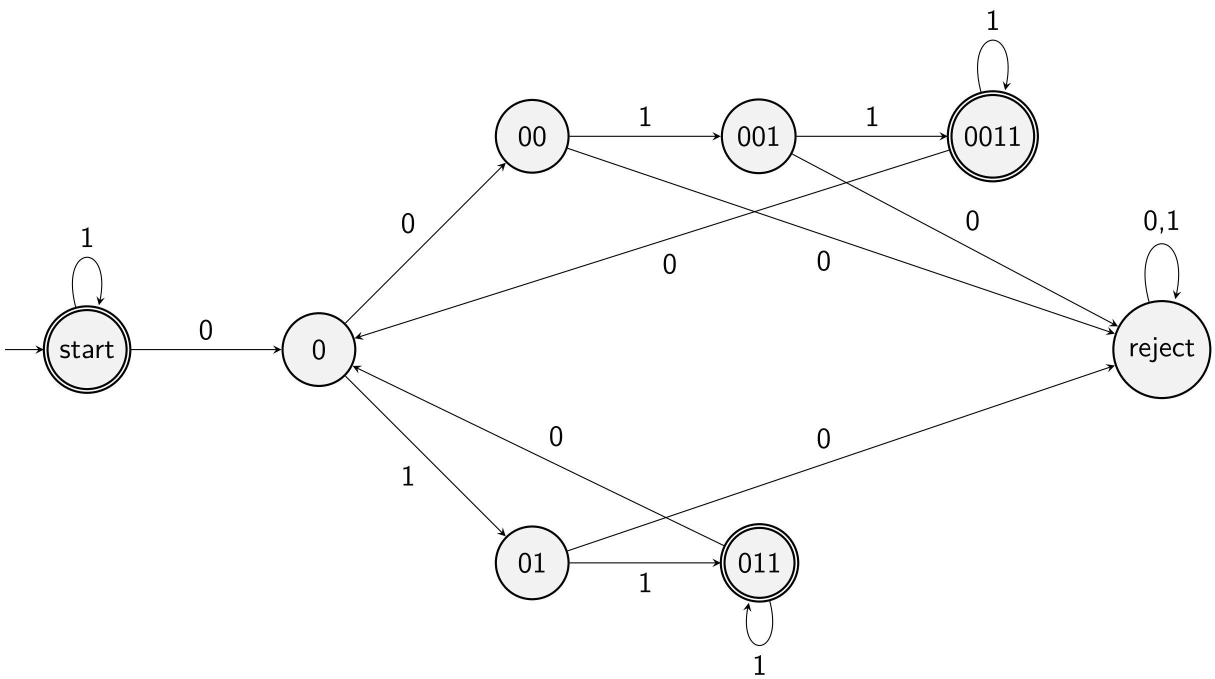
Solution:

Solution 1:



Alternate solution on next page.

Solution 2:

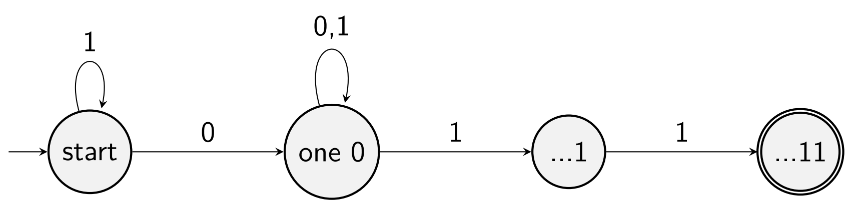


There may be other valid solutions

(d) Let K be the language containing all binary strings x such that both of the following are true:
- x contains at least one 0 **and**
- x ends with 11

Draw an *NFA* that recognizes K.

Solution:



9. Irregularity/Uncountability

Do exactly **one** of these two problems.

I am doing
◯ part (a) Irregularity
◯ part (b) Uncountability

(a) Prove that the language $L = \{10^x 10^{x+1} 1 : x \geq 0\}$ is not regular.

Solution:

Suppose for contradiction there exists a DFA D that recognizes L.

Consider the set $S = \{10^n 1 : n \in \mathbb{N}\}$. Since S contains infinitely many strings and D has a finite number of states, there must be two distinct strings $10^i 1 \in S$ and $10^j 1 \in S$ for some $i \neq j$ that end at the same state in D.

Now, append $0^{i+1}1$ to both strings. The resulting strings are:

$x = 10^i 10^{i+1} 1 \in L$ and $y = 10^j 10^{i+1} 1 \notin L$ since $i \neq j$.

Both x and y end up at the same state of D, call it q. Since $x \in L$, state q must be an accept state, but then D would incorrectly accept $y \notin L$. So D does not recognize L.

Thus, no DFA recognizes L, and L is irregular.

(b) Olympic champion, Alysa Liu, is so incredible that her next free skate will contain infinitely many jumps. She can do single, double, or triple versions of the following 6 jumps: axel, salchow, loop, flip, toe-loop, and lutz. Prove that the set of free skates Alysa can choose from is uncountable.

Solution:

Let A be the set of all of Alysa's possible free skates. Suppose for the sake of contradiction that A is countable. Then there exists a listing of elements of A: $a_1, a_2, a_3, \ldots$.

Let $a_{i,j}$ denote the jth jump in the free skate a_i. Define a new free skate a_{diag} by

$$a_{diag,i} = \begin{cases} \text{double axel} & \text{if } a_{i,i} \neq \text{double axel} \\ \text{triple flip} & \text{if } a_{i,i} = \text{double axel} \end{cases}$$

For all i, we have $a_{diag,i} \neq a_{i,i}$. Thus a_{diag} differs from a_i in the ith jump for every i. Therefore $a_{diag} \neq a_i$ for any i, meaning the list is incomplete. This contradicts the assumption that all elements of A were listed.

Thus, the set of all free skates that Alysa can perform is uncountable.

10. Undecidability

(a) Consider the following problem: Given an arbitrarily large number n and an integer x, output whether or not n has exactly x prime factors.

Is this problem decidable? Provide a brief justification for your answer.

Solution:

Yes, this problem is decidable. In the worst case, to find all factors of n, we could loop through all numbers i from 1 to n and check if i divides n. If i divides n, to check if i is prime, we could loop through all numbers j from 2 to $i-1$ and check if j divides i. If no such j divides i, then i is prime. Once we find all of the prime factors of n, we can count them and see if there are x of them. This algorithm will **always** halt and output the correct answer.

Note that while this problem is decidable, for sufficiently large values of n, even the most efficient known algorithms are extremely slow and would not finish running before the sun explodes.

(b) Consider the following problem: Given CODE(P), determine whether $P(x)$ produces the same output as $P(\text{reverse}(x))$ for every input x.

Is this problem decidable? Provide a brief justification for your answer.

Solution:

It is undecidable. Whether $P(x) = P(\text{reverse}(x))$ for every input x is a non-trivial property of P's behavior (some programs have this property, some do not), so by Rice's Theorem it is undecidable.

11. Bonus Problem: Boolean Algebra

Let f be the boolean function defined as $f(x, y, z) = (x + y)' + (zy)$

(a) Fill in the following table with the values of $f(x, y, z)$ in the last column. Feel free to use the blank columns while doing your work.

x	y	z				$f(x, y, z)$
0	0	0				
0	0	1				
0	1	0				
0	1	1				
1	0	0				
1	0	1				
1	1	0				
1	1	1				

Solution:

x	y	z	$(x+y)$	$(x+y)'$	(zy)	$f(x,y,z)$
0	0	0	0	1	0	1
0	0	1	0	1	0	1
0	1	0	1	0	0	0
0	1	1	1	0	1	1
1	0	0	1	0	0	0
1	0	1	1	0	0	0
1	1	0	1	0	0	0
1	1	1	1	0	1	1

(b) Write $f(x, y, z)$ as a Boolean algebra expression in the canonical sum-of-products (DNF) form.

Solution:

$f(x, y, z) = x'y'z' + x'y'z + x'yz + xyz$

(c) Write $f(x, y, z)$ as a Boolean algebra expression in the canonical product-of-sums (CNF) form.

Solution:

$f(x, y, z) = (x + y' + z)(x' + y + z)(x' + y + z')(x' + y' + z)$