

CSE 390Z: Mathematics for Computation Workshop

Mid-Quarter Review Solutions

Name: _____

1. Predicate Translation

Let the domain of discourse be novels, comic books, movies, and TV shows. Translate the following statements to predicate logic, using the following predicates:

$\text{Novel}(x) := x$ is a novel

$\text{Comic}(x) := x$ is a comic book

$\text{Movie}(x) := x$ is a movie

$\text{Show}(x) := x$ is a TV show

$\text{Adaptation}(x, y) := x$ is an adaptation of y

You may use $=$ as a predicate to test if two things are the same.

- (a) Every movie is an adaptation of a novel or a comic book.

Solution:

$$\forall m(\text{Movie}(m) \rightarrow \exists x[(\text{Novel}(x) \vee \text{Comic}(x)) \wedge \text{Adaptation}(m, x)])$$

- (b) Every movie is an adaptation of exactly one novel.

Solution:

$$\forall m(\text{Movie}(m) \rightarrow \exists x(\text{Novel}(x) \wedge \text{Adaptation}(m, x) \wedge \forall y[(\text{Novel}(y) \wedge \text{Adaptation}(m, y)) \rightarrow (y = x)]))$$

OR

$$\forall m(\text{Movie}(m) \rightarrow \exists x(\text{Novel}(x) \wedge \text{Adaptation}(m, x) \wedge \forall y[(y \neq x) \rightarrow \neg(\text{Novel}(y) \wedge \text{Adaptation}(m, y))]))$$

OR

$$\forall m(\text{Movie}(m) \rightarrow \exists x \forall y[(y = x) \leftrightarrow (\text{Novel}(y) \wedge \text{Adaptation}(m, y))])$$

- (c) Using the same domain of discourse and predicates as above, translate the following predicate logic statement into English. Your translation should be as natural as possible.

$$\forall x(\text{Novel}(x) \rightarrow (\neg \exists m[\text{Adaptation}(m, x) \wedge \text{Movie}(m)] \vee \neg \exists s[\text{Adaptation}(s, x) \wedge \text{Show}(s)]))$$

Solution:

A novel cannot be adapted into both a movie and a TV show.

2. Normal Forms

Consider the following function F :

| p | q | r | $F(p, q, r)$ |
|-----|-----|-----|--------------|
| T | T | T | F |
| T | T | F | F |
| T | F | T | T |
| T | F | F | F |
| F | T | T | T |
| F | T | F | F |
| F | F | T | F |
| F | F | F | T |

- (a) Write a propositional logic expression for F in DNF form (ORs of ANDs).

Solution:

$$(p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$$

- (b) Write a propositional logic expression for F in CNF form (ANDs of ORs).

Solution:

$$(\neg p \vee \neg q \vee \neg r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee q \vee \neg r)$$

3. Modular Arithmetic

Write an **English proof** that for all integers $x, y, n > 0$, if $x \equiv_{6n} 1$ and $y \equiv_{7n} 5$ then $7x + 2y \equiv_{14n} 17$.

Hint: Apply the definition of congruence and divides.

Solution:

Let $x, y, n > 0$ be arbitrary integers. Suppose $x \equiv_{6n} 1$ and $y \equiv_{7n} 5$. Then by definition of congruence, $6n \mid (x - 1)$ and $7n \mid (y - 5)$. Then by definition of divides, there exists integers j, k such that $x - 1 = 6nk$ and $y - 5 = 7nj$. Thus $x = 6nk + 1$ and $y = 7nj + 5$. Then we have:

$$\begin{aligned} 7x + 2y &= 7(6nk + 1) + 2(7nj + 5) \\ &= 42nk + 7 + 14nj + 10 \\ &= 42nk + 14nj + 17 \\ &= 14n(3k + j) + 17 \end{aligned}$$

We have shown that $(7x + 2y) - 17 = 14n(3k + j)$. So by definition of divides, $14n \mid (7x + 2y) - 17$. Then by definition of congruence, $7x + 2y \equiv_{14n} 17$. Since x, y, n were arbitrary, the claim holds.

4. Extended Euclidean Algorithm

Solve the equation and state the full set of solutions

$$31y \equiv_{83} 2$$

Solution:

First we use the Euclidean algorithm to compute $\gcd(83, 31)$ keeping track of our equations in every step

| | |
|--|---------------------------|
| $\gcd(83, 31) = \gcd(31, 83 \% 31) = \gcd(31, 21)$ | $83 = 2 * 31 + 21$ |
| $\gcd(31, 21) = \gcd(21, 31 \% 21) = \gcd(21, 10)$ | $31 = 1 * 21 + 10$ |
| $\gcd(21, 10) = \gcd(10, 21 \% 10) = \gcd(10, 1)$ | $21 = 2 * 10 + 1$ |
| $\gcd(10, 1) = \gcd(1, 10 \% 1) = \gcd(1, 0) = 1$ | no equation for this line |

Now, we rearrange:

| | | |
|--------------------|--------------------|-----|
| $83 = 2 * 31 + 21$ | $21 = 83 - 2 * 31$ | (1) |
| $31 = 1 * 21 + 10$ | $10 = 31 - 1 * 21$ | (2) |
| $21 = 2 * 10 + 1$ | $1 = 21 - 2 * 10$ | (3) |

Now we use back substitution to find an equation of the form $\gcd(83, 31) = s * 83 + t * 31$.

The labels used below are from the previous step.

| | |
|---------------------------------|-------------------------|
| $1 = 21 - 2 * 10$ | Start with equation (3) |
| $= 21 - 2 * (31 - 1 * 21)$ | Sub in equation (2) |
| $= 21 - 2 * 31 + 2 * 21$ | |
| $= -2 * 31 + 3 * 21$ | |
| $= -2 * 31 + 3 * (83 - 2 * 31)$ | Sub in equation (1) |
| $= -2 * 31 + 3 * 83 - 6 * 31$ | |
| $= 3 * 83 - 8 * 31$ | |

So -8 is our multiplicative inverse. Since we want an inverse j such that $0 \leq j < 83$ we use $-8 + 83 = 75$ as our multiplicative inverse.

We multiply both sides of our original equation $31y \equiv_{83} 2$ by 75.

$$\begin{aligned} 75 \cdot 31y &\equiv_{83} 75 \cdot 2 \\ y &\equiv_{83} 150 \equiv_{83} 67 \end{aligned}$$

So the full set of solutions is $67 + 83k$ for any integer k .

5. Induction

Prove by induction that $3^n - 1$ is divisible by 2 for any integer $n \geq 1$.

Solution:

1. Let $P(n)$ be the statement " $3^n - 1$ is divisible by 2". We prove $P(n)$ for all integers $n \geq 1$ by induction.
2. Base Case ($n = 1$): $3^n - 1 = 3^1 - 1 = 3 - 1 = 2 = 2 \cdot 1$. By definition of divides, $2 \mid 2$. So $P(1)$ holds.
3. Inductive Hypothesis:
Suppose that $P(k)$ holds for some arbitrary integer $k \geq 1$. Then $2 \mid 3^k - 1$. Then by definition of divides, there exists some integer a such that $3^k - 1 = 2a$.
4. Inductive Step: Goal: Prove $P(k+1)$ i.e., prove that $2 \mid (3^{k+1} - 1)$.

We have:

$$\begin{aligned} 3^{k+1} - 1 &= 3(3^k) - 1 \\ &= 3(3^k - 1 + 1) - 1 && \text{Creatively add 0 (0 = -1+1)} \\ &= 3(2a + 1) - 1 && \text{By IH} \\ &= 6a + 3 - 1 \\ &= 6a + 2 \\ &= 2(3a + 1) \end{aligned}$$

Thus by definition of divides, $2 \mid 3^{k+1} - 1$. So $P(k+1)$ holds.

5. We have shown that $P(n)$ holds for all integers $n \geq 1$ by induction.

6. Strong Induction

Consider the function f , which takes a natural number as input and outputs a natural number.

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ f(n-1) + 2 \cdot f(n-2) & \text{if } n \geq 2 \end{cases}$$

Prove that $f(n) = 2^n$ for all $n \in \mathbb{N}$.

Solution:

1. Let $P(n)$ be the claim that $f(n) = 2^n$. We will prove $P(n)$ holds for all $n \in \mathbb{N}$ by strong induction.

2. Base Cases ($n = 0, n = 1$):

$f(0) = 1 = 1 = 2^0$ so $P(0)$ holds.

$f(1) = 2 = 2 = 2^1$ so $P(1)$ holds.

3. Inductive Hypothesis:

Option 1: Suppose that $P(j)$ holds for all $j \in \mathbb{N}$ such that $j \leq k$ for some arbitrary integer $k \geq 1$.

Option 2: Suppose that $P(0) \wedge P(1) \wedge \dots \wedge P(k)$ for some arbitrary integer $k \geq 1$.

4. Inductive Step: Goal: Show $P(k+1)$ i.e., show that $f(k+1) = 2^{k+1}$

Since $k \geq 1$, $k+1 \geq 2$, so we are guaranteed to be in the recursive part of $f(n)$. We have:

$$\begin{aligned} f(k+1) &= f(k+1-1) + 2 \cdot f(k+1-2) && \text{Definition of } f \\ &= f(k) + 2 \cdot f(k-1) \\ &= 2^k + 2 \cdot 2^{k-1} && \text{I.H. twice} \\ &= 2^k + 2^{k-1+1} \\ &= 2^k + 2^k \\ &= 2 \cdot 2^k \\ &= 2^{k+1} \end{aligned}$$

So, $P(k+1)$ holds.

5. Therefore, we have proven that the claim holds for all $n \in \mathbb{N}$ by strong induction.

