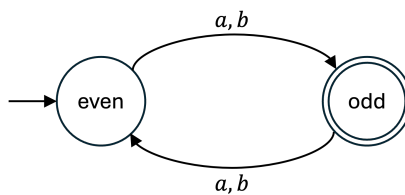


### 0. Constructing DFAs

For each of the following, construct a DFA for the specified language over the alphabet  $\Sigma = \{a, b\}$ .

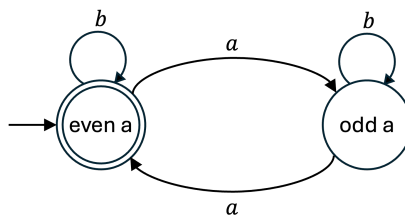
(a) Strings with odd length.

**Solution:**



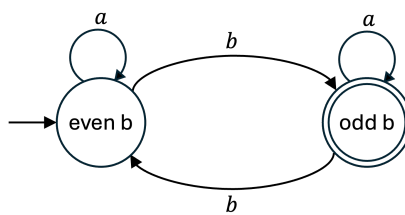
(b) Strings with an even number of  $a$ 's.

**Solution:**



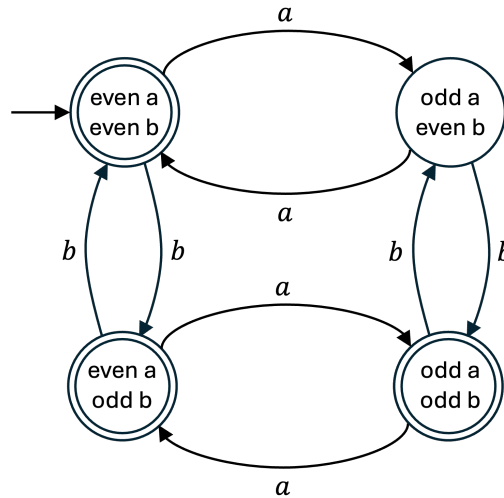
(c) Strings with an odd number of  $b$ 's.

**Solution:**



(d) Strings with an even number of  $a$ 's **or** an odd number of  $b$ 's.

**Solution:**



**1. All the Things**

Let  $\Sigma := \{0, 1, 2, 3, 4, 5\}$ . For an arbitrary string  $x$  over  $\Sigma$ , we can write  $x = x_0x_1 \cdots x_n$ , where  $x_0, x_1, \dots, x_n \in \Sigma$ . Define a language  $L$  over  $\Sigma$  as follows:

$x \in L$  iff for every position  $i$  from 0 to  $n$ , if the value of  $x_i$  is odd, then every digit (character) that comes after  $x_i$  must be **greater** than  $x_i$ .

For example, the string  $2124 \in L$  because 1 is the only odd digit and every digit after 1 is greater than 1.

The string  $21254 \notin L$  because 5 is an odd digit, 4 comes after 5, and  $4 < 5$ .

The string  $211 \notin L$  because 1 comes after 1 and  $1 \not> 1$ .

(a) List 3 strings in  $L$  and 3 strings not in  $L$ . The strings should be over the alphabet  $\Sigma$ .

**Solution:**

**Accepted:**

- 145
- 135
- 12425
- 2004
- 2034

**Rejected:**

- 321
- 11
- 455
- 452
- 2010

(b) Construct a regular expression for the language  $L$ .

**Solution:**

$$(0 \cup 2 \cup 4)^*(\epsilon \cup 1)(2 \cup 4)^*(\epsilon \cup 3)4^*(\epsilon \cup 5)$$

(c) Construct a CFG for the language  $L$ .

**Solution:**

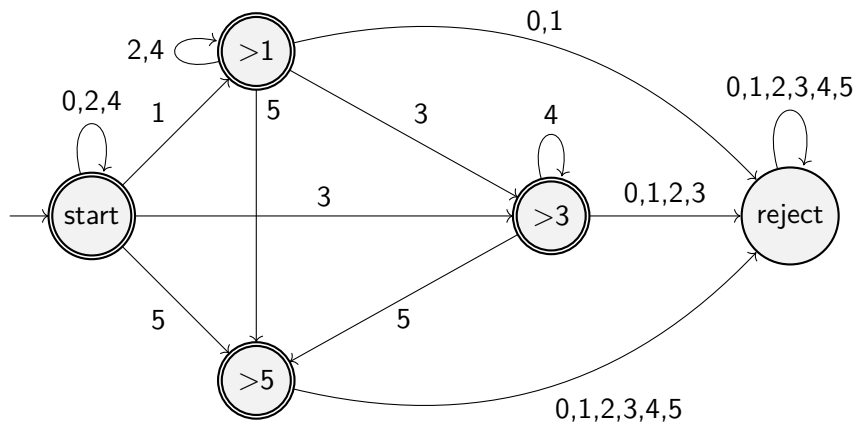
- S**  $\rightarrow$  **ABCDEF**
- A**  $\rightarrow$  **AA** | 0 | 2 | 4 |  $\epsilon$
- B**  $\rightarrow$  1 |  $\epsilon$
- C**  $\rightarrow$  **CC** | 2 | 4 |  $\epsilon$
- D**  $\rightarrow$  3 |  $\epsilon$
- E**  $\rightarrow$  **EE** | 4 |  $\epsilon$
- F**  $\rightarrow$  5 |  $\epsilon$

Alternate Solution:

- S**  $\rightarrow$  0**S** | 2**S** | 4**S** | **A**
- A**  $\rightarrow$  1**B** | **B**
- B**  $\rightarrow$  2**B** | 4**B** | **C**
- C**  $\rightarrow$  3**D** | **D**
- D**  $\rightarrow$  4**D** | **E**
- E**  $\rightarrow$  5 |  $\epsilon$

(d) Construct a DFA for the language  $L$ .

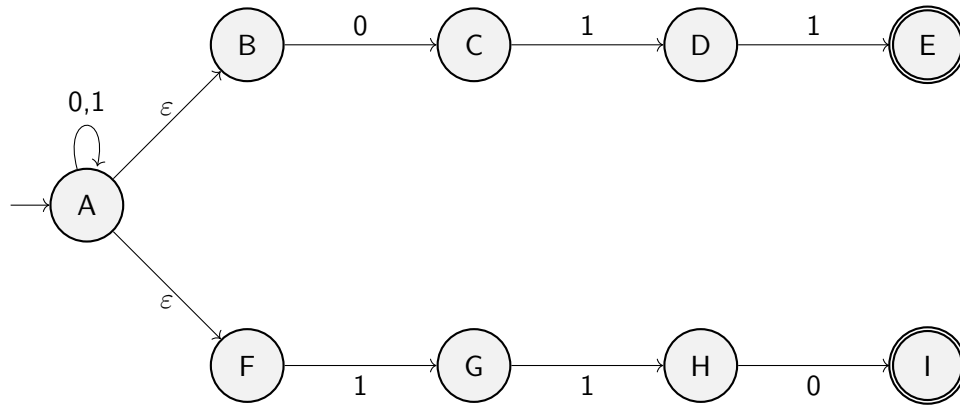
**Solution:**



**1. NFA to DFA**

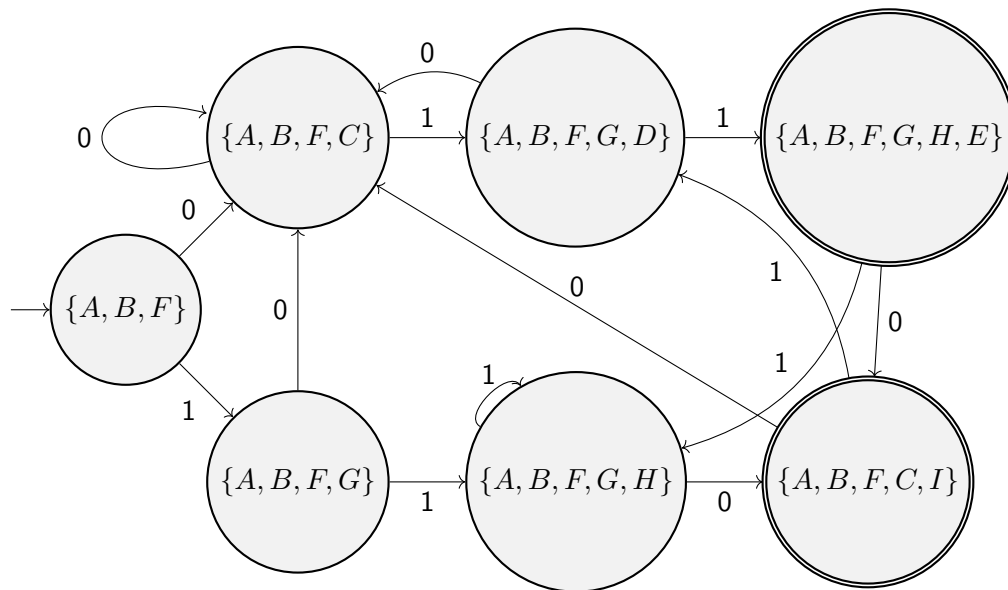
(a) Construct an NFA for the language "all binary strings ending in either 011 or 110".

**Solution:**



(b) Use the technique you saw in 311 lecture to construct an equivalent DFA for the same language.

**Solution:**

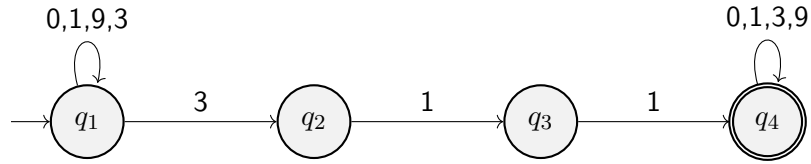


## 2. NFA to DFA, DFA Minimization

Let  $\Sigma = \{0, 1, 3, 9\}$ . Let  $L$  be the language over  $\Sigma$  that contains all strings that have "311" as a substring.

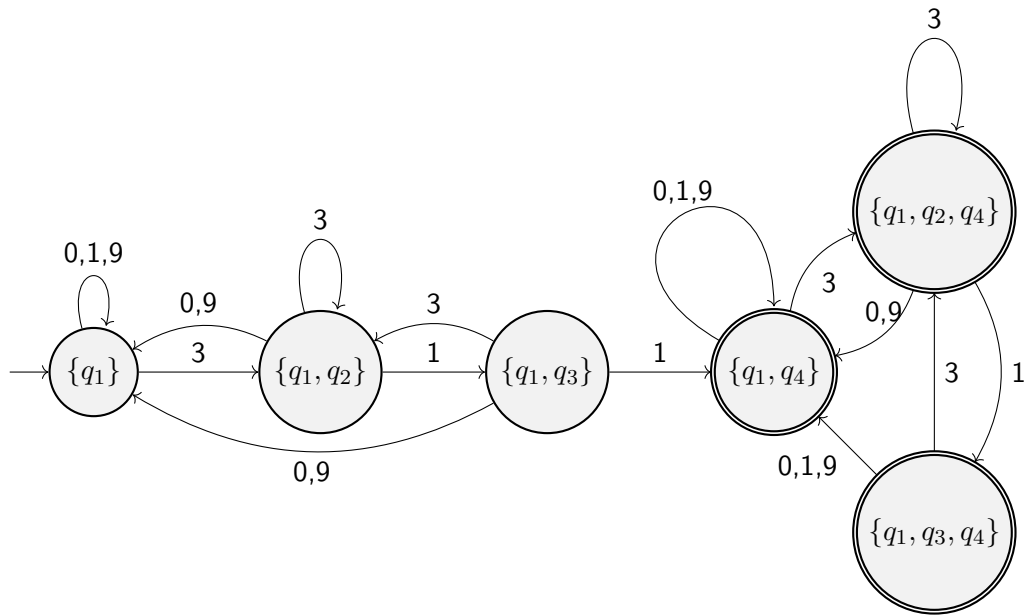
(a) Give an NFA to accept strings in  $L$ .

**Solution:**



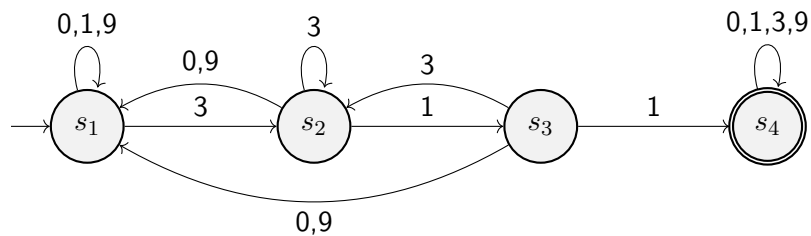
(b) Give an equivalent DFA for your NFA (using the algorithm from 311).

**Solution:**



(c) Is your DFA minimized? If not, give the minimized DFA using the algorithm from 311.

**Solution:**

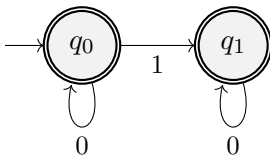


### 3. More NFAs

- (a) Construct an NFA for the language "all strings from the alphabet  $\Sigma = \{0, 1, 2\}$  containing only 0's and 1's, and at most one 1".

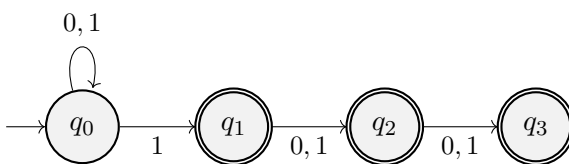
For instance, the strings 0000, 0010, 1000, 0, 1, and  $\epsilon$  should be accepted. The strings 0101, 2, 000020, 102000, 011, should be rejected.

**Solution:**



- (b) Construct an NFA for the language "all binary strings that have a 1 as one of the last three digits".

**Solution:**

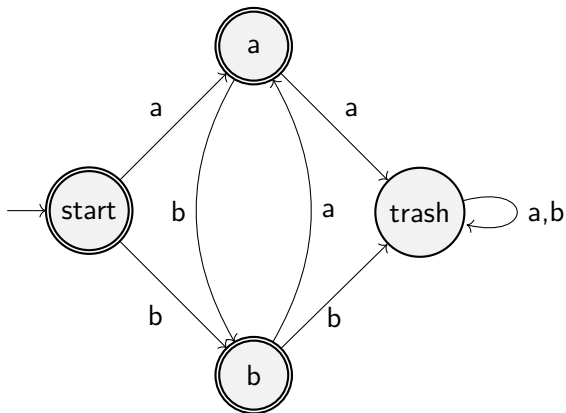


## 4. Seeing Double

Consider the language  $L$  of strings over the alphabet  $\Sigma := \{a, b\}$  with alternating  $a$ 's and  $b$ 's (i.e., not containing  $aa$  or  $bb$ ).

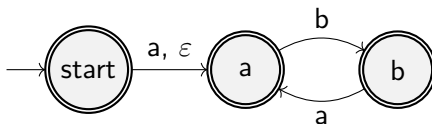
(a) Construct a DFA that recognizes  $L$ .

**Solution:**



(b) Construct an NFA that recognizes  $L$ .

**Solution:**



## 5. Irregular Languages

(a) Let  $L = \{0^m 1^{m+2} 0\}$ . Prove that  $L$  is irregular

### Solution:

Suppose for contradiction there exists some DFA  $M$  that recognizes  $L$ .

Consider the set  $S = \{0^m : m \in \mathbb{N}\}$ . Since  $S$  is infinite, and  $M$  has finitely many states, there must be two distinct strings,  $0^a$  and  $0^b$  for some  $a \neq b$  that end at the same state in  $M$ .

Consider appending  $1^{a+2}0$  to both strings.

Note that  $0^a 1^{a+2} 0 \in L$ , but  $0^b 1^{a+2} 0 \notin L$  since  $a \neq b$ , but they both end up in the same state of  $M$ , call it  $q$ . Since  $0^a 1^{a+2} 0 \in L$ , state  $q$  must be an accept state, but then  $M$  would incorrectly accept  $0^b 1^{a+2} 0 \notin L$ , so  $M$  does not recognize  $L$ .

Thus, no DFA recognizes  $L$ , and  $L$  is irregular.

(b) Let  $L = \{0^m 1^n : m, n \in \mathbb{N}, m = 2n\}$ . Prove that  $L$  is irregular

### Solution:

Suppose for contradiction there exists some DFA  $M$  that recognizes  $L$ .

Consider the set  $S = \{0^{2k} : k \in \mathbb{N}\}$ . Since  $S$  is infinite, and  $M$  has finitely many states, there must be two distinct strings,  $0^{2a}$  and  $0^{2b}$  for some  $a \neq b$  that end at the same state in  $M$ .

Consider appending  $1^a$  to both strings.

Note that  $0^{2a} 1^a \in L$ , but  $0^{2b} 1^a \notin L$  since  $a \neq b$ , but they both end up in the same state of  $M$ , call it  $q$ . Since  $0^{2a} 1^a \in L$ , state  $q$  must be an accept state, but then  $M$  would incorrectly accept  $0^{2b} 1^a \notin L$ , so  $M$  does not recognize  $L$ .

Thus, no DFA recognizes  $L$ , and  $L$  is irregular.

## 6. CARDinality

So glad you decided to play CARDinality! CARDinality is played with a standard 52 card deck. The rules are simple. A game of CARDinality consists of an infinite sequence of moves. A move means playing a single card. Prove that the set of all CARDinality games is uncountable.

### Solution:

Let  $C$  be the set of all CARDinality games. Suppose for the sake of contradiction that  $C$  is countable. Then there exists a listing of elements of  $C$   $c_0, c_1, c_2, \dots$

We use  $c_{i,j}$  to denote the  $j$ th move in game  $c_i$ . Note that we're 0-indexing the moves so move 0 is really the first move in the game. Define a new game  $c_{diag}$  by:

$$c_{diag,i} = \begin{cases} 2 \text{ of diamonds} & \text{if } c_{i,i} \neq 2 \text{ of diamonds} \\ \text{ace of spades} & \text{if } c_{i,i} = 2 \text{ of diamonds} \end{cases}$$

For all  $i$ , we have  $c_{diag,i} \neq c_{i,i}$ . In other words,  $c_{diag}$  makes a different  $i$ th move than  $c_i$  for all  $i$ . Therefore,  $c_{diag} \neq c_i$  for any  $i$  and the list is incomplete. Thus, the set of all CARDinality games is uncountable.

## 7. Is That Even a Word?

Let  $\Sigma = \{a, b, c, \dots, y, z\}$ . In other words,  $\Sigma$  is the set of all lowercase English letters. Prove that the set of functions  $f : \Sigma^* \rightarrow \{0, 1\}$  is uncountable.

### Solution:

Let  $F$  be the set of all functions  $f : \Sigma^* \rightarrow \{0, 1\}$ . Suppose for contradiction that  $F$  is countable. Then there exists a listing of all functions in  $F$   $f_0, f_1, f_2, \dots$

We create a list of all elements in  $\Sigma^*$  sorted in lexicographic order and use  $w_j$  to denote the  $j$ th word in this list. Construct a new function  $d : \Sigma^* \rightarrow \{0, 1\}$  as follows:

$$d(w_i) = \begin{cases} 0 & \text{if } f_i(w_i) = 1 \\ 1 & \text{if } f_i(w_i) = 0 \end{cases}$$

For all  $i$ , we have  $d(w_i) \neq f_i(w_i)$ . Therefore,  $d \neq f_i$  for any  $i$  and the list is incomplete. Thus, the set of all functions  $f : \Sigma^* \rightarrow \{0, 1\}$  is uncountable.