CSE 390Z: Mathematics for Computation Workshop

Week 7 Workshop Solutions

0. Structural Induction: Divisible by 4

Define a set T of numbers by:

- 4 and 12 are in T
- If $x \in T$ and $y \in T$, then $x + y \in T$ and $x - y \in T$

Prove by structural induction that every number in T is divisible by 4.

Solution:

Let $P(b)$ be the claim that $4 \mid b$. We will prove $P(b)$ is true for all numbers $b \in T$ by structural induction.
Base Case:

- $4 = 1 \cdot 4$, so $4|4$ and $P(4)$ holds.
- $12 = 3 \cdot 4$, so $4 \mid 12$ and $P(12)$ holds.

Inductive Hypothesis: Suppose $P(x)$ and $P(y)$ for some arbitrary $x, y \in T$.
Inductive Step:

Goal: Prove $P(x+y)$ and $P(x-y)$

Per the IH, $4 \mid x$ and $4 \mid y$. By the definition of divides, $x = 4k$ and $y = 4j$ for some integers k, j.

Goal: Show P(x+y)
$x + y = 4k + 4j = 4(k + j)$. By definition of divides, $4 \mid x + y$ and $P(x + y)$ holds.

Goal: Show P(x-y)
Similarly, $x - y = 4k - 4j = 4(k - j)$. By the definition of divides, $4 \mid x - y$ and $P(x - y)$ holds.

Conclusion: Therefore, $P(b)$ holds for all numbers $b \in T$.

1. More Induction...Literally

Define a set S as follows:
Basis: $6 \in S$; $15 \in S$
Recursive: if $x, y \in S$ then $x + y \in S$

Define a set T as follows:
Basis: $6 \in T$; $15 \in T$
Recursive: if $x \in T$ then $x+6 \in T$ and $x+15 \in T$

In lecture you proved that every element of T is an element of S.
Now we're going to prove that every element of S is an element of T.

(a) First, use structural induction to prove the following lemma:
The sum of any two elements in T is also in T. Formally this is: $\forall a, b \in T(a + b \in T)$

Solution:

Let $P(b)$ be "$a + b \in T$ for all $a \in T$". We prove $P(b)$ for all $b \in T$ by structural induction.

Base Case:

$(b = 6)$: Let $a \in T$ be arbitrary. $a + b = a + 6 \in T$ by the recursive step. So $P(6)$ holds.

$(b = 15)$: Let $a \in T$ be arbitrary. $a + b = a + 15 \in T$ by the recursive step. So $P(15)$ holds.

Inductive Hypothesis: Assume that $P(b)$ is true for some arbitrary $b \in T$. i.e., assume that for all $a \in T$, $a + b \in T$.

Inductive Step: We need to show $P(b + 6)$ and $P(b + 15)$.

Goal: Show P(b+6): Let $a \in T$ be arbitrary. $a + (b + 6) = (a + b) + 6$. From the inductive hypothesis, we know $a + b \in T$. Therefore, by the recursive step, $(a + b) + 6 \in T$. Since a was arbitrary, we have shown $P(b + 6)$.

Goal: Show P(b+15): Let $a \in T$ be arbitrary. $a + (b + 15) = (a + b) + 15$. From the inductive hypothesis, we know $a + b \in T$. Therefore, by the recursive step, $(a + b) + 15 \in T$. Since a was arbitrary, we have shown $P(b + 15)$.

We have shown the claim holds for all $b \in T$ by induction.

(b) Now, use structural induction to prove the main claim: Every element of S is also in T.
You can use the Lemma from part (a) by citing "part (a) lemma".

Solution:

Let $P(x)$ be "$x \in T$". We prove $P(x)$ is true for all $x \in S$ by structural induction.

Base Case: $6 \in T$ and $15 \in T$, both by the basis step, so $P(6)$ and $P(15)$ are true.

Inductive Hypothesis: Suppose that $P(x)$ and $P(y)$ are true for some arbitrary $x, y \in S$.

Inductive Step: We need to show that $P(x + y)$ holds. By the inductive hypothesis, we know $P(x)$ and $P(y)$ hold i.e., $x \in T$ and $y \in T$. By the lemma from part (a), we can conclude that $x + y \in T$, so $P(x + y)$ holds.

Therefore, $P(x)$ is true for all $x \in S$ by induction.

2. Structural Induction: CharTrees

Recursive Definition of CharTrees:

- Basis Step: `Null` is a **CharTree**
- Recursive Step: If L, R are **CharTrees** and $c \in \Sigma$, then $\texttt{CharTree}(L, c, R)$ is also a **CharTree**

Intuitively, a **CharTree** is a tree where the non-null nodes store a `char` data element.

Recursive functions on CharTrees:

- The preorder function returns the preorder traversal of all elements in a **CharTree**.

$$\begin{aligned} \text{preorder}(\texttt{Null}) &= \varepsilon \\ \text{preorder}(\texttt{CharTree}(L, c, R)) &= c \cdot \text{preorder}(L) \cdot \text{preorder}(R) \end{aligned}$$

- The postorder function returns the postorder traversal of all elements in a **CharTree**.

$$\begin{aligned} \text{postorder}(\texttt{Null}) &= \varepsilon \\ \text{postorder}(\texttt{CharTree}(L, c, R)) &= \text{postorder}(L) \cdot \text{postorder}(R) \cdot c \end{aligned}$$

- The mirror function produces the mirror image of a **CharTree**.

$$\begin{aligned} \text{mirror}(\texttt{Null}) &= \texttt{Null} \\ \text{mirror}(\texttt{CharTree}(L, c, R)) &= \texttt{CharTree}(\text{mirror}(R), c, \text{mirror}(L)) \end{aligned}$$

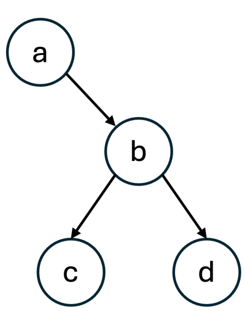
- Finally, for all strings x, x^R, the "reversal" of x, produces the string in reverse order.

Additional Facts:
You may use the following facts:

- **Fact 1:** For any strings $x_1, ..., x_k$: $(x_1 \cdot ... \cdot x_k)^R = x_k^R \cdot ... \cdot x_1^R$
- **Fact 2:** For any character c, $c^R = c$

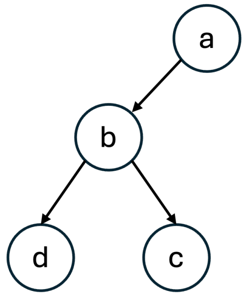
It turns out that for any CharTree T, the reversal of the preorder traversal of T is the same as the postorder traversal of the mirror of T.

Example for Intuition:



preorder$(T) =$"abcd".
T is built as $(\texttt{Null}, a, U)$
Where U is (V, b, W),
$V = (\texttt{Null}, c, \texttt{Null}), W = (\texttt{Null}, d, \texttt{Null})$.

Let T be the tree above.



This tree is mirror(T).
postorder(mirror(T)) =“dcba”,
“dcba” is the reversal of “abcd” so
$[\text{preorder}(T)]^R = \text{postorder}(\text{mirror}(T))$ holds for T

Use structural induction to prove the following claim:

For every **CharTree**, T: $[\text{preorder}(T)]^R = \text{postorder}(\text{mirror}(T))$

Solution:

Let $P(T)$ be “$[\text{preorder}(T)]^R = \text{postorder}(\text{mirror}(T))$”. We show $P(T)$ holds for all **CharTree**s T by structural induction.

Base case ($T = \texttt{Null}$):
LHS: $[\text{preorder}(\texttt{Null})]^R = \varepsilon^R = \varepsilon$
RHS: $\text{postorder}(\text{mirror}(T)) = \text{postorder}(\texttt{Null}) = \varepsilon$
Since LHS = RHS, $P(\texttt{Null})$ holds.

Inductive hypothesis: Suppose $P(L), P(R)$ both holdfor arbitrary **CharTree**s L, R.

Inductive step:
Let $T = \text{CharTree}(L, c, R)$ for an arbitrary $c \in \Sigma$. We want to show $P(T)$ i.e.,
$[\text{preorder}(\text{CharTree}(L, c, R))]^R = \text{postorder}(\text{mirror}(\text{CharTree}(L, c, R)))$.

$$
\begin{aligned}
[\text{preorder}(T)]^R &= [\text{preorder}(\text{CharTree}(L, c, R))]^R && \text{Def of } T \\
&= [c \cdot \text{preorder}(L) \cdot \text{preorder}(R)]^R && \text{Def of preorder} \\
&= \text{preorder}(R)^R \cdot \text{preorder}(L)^R \cdot c^R && \text{Fact 1} \\
&= \text{preorder}(R)^R \cdot \text{preorder}(L)^R \cdot c && \text{Fact 2} \\
&= \text{postorder}(\text{mirror}(R)) \cdot \text{postorder}(\text{mirror}(L)) \cdot c && \text{By I.H.} \\
&= \text{postorder}(\text{CharTree}(\text{mirror}(R), c, \text{mirror}(L)) && \text{Def of postorder} \\
&= \text{postorder}(\text{mirror}(\text{CharTree}(L, c, R))) && \text{Def of mirror} \\
&= \text{postorder}(\text{mirror}(T)) && \text{Def of } T
\end{aligned}
$$

So $P(\texttt{CharTree}(L, c, R))$ holds.
By the principle of induction, $P(T)$ holds for all **CharTree**s T.

3. More Induction...Literally

Define a set S as follows:
Basis: $6 \in S$; $15 \in S$
Recursive: if $x, y \in S$ then $x + y \in S$

Define a set T as follows:
Basis: $6 \in T$; $15 \in T$
Recursive: if $x \in T$ then $x+6 \in T$ and $x+15 \in T$

In lecture you proved that every element of T is an element of S.
Now we're going to prove that every element of S is an element of T.

(a) First, use structural induction to prove the following lemma:
The sum of any two elements in T is also in T. Formally this is: $\forall a, b \in T(a + b \in T)$

Solution:

Let $P(b)$ be "$a + b \in T$ for all $a \in T$". We prove $P(b)$ for all $b \in T$ by structural induction.

Base Case:

$(b = 6)$: Let $a \in T$ be arbitrary. $a + b = a + 6 \in T$ by the recursive step. So $P(6)$ holds.

$(b = 15)$: Let $a \in T$ be arbitrary. $a + b = a + 15 \in T$ by the recursive step. So $P(15)$ holds.

Inductive Hypothesis: Assume that $P(b)$ is true for some arbitrary $b \in T$. i.e., assume that for all $a \in T$, $a + b \in T$.

Inductive Step: We need to show $P(b + 6)$ and $P(b + 15)$.

Goal: Show P(b+6): Let $a \in T$ be arbitrary. $a + (b + 6) = (a + b) + 6$. From the inductive hypothesis, we know $a + b \in T$. Therefore, by the recursive step, $(a + b) + 6 \in T$. Since a was arbitrary, we have shown $P(b + 6)$.

Goal: Show P(b+15): Let $a \in T$ be arbitrary. $a + (b + 15) = (a + b) + 15$. From the inductive hypothesis, we know $a + b \in T$. Therefore, by the recursive step, $(a + b) + 15 \in T$. Since a was arbitrary, we have shown $P(b + 15)$.

We have shown the claim holds for all $b \in T$ by induction.

(b) Now, use structural induction to prove the main claim: Every element of S is also in T.
You can use the Lemma from part (a) by citing "part (a) lemma".

Solution:

Let $P(x)$ be "$x \in T$". We prove $P(x)$ is true for all $x \in S$ by structural induction.

Base Case: $6 \in T$ and $15 \in T$, both by the basis step, so $P(6)$ and $P(15)$ are true.

Inductive Hypothesis: Suppose that $P(x)$ and $P(y)$ are true for some arbitrary $x, y \in S$.

Inductive Step: We need to show that $P(x + y)$ holds. By the inductive hypothesis, we know $P(x)$ and $P(y)$ hold i.e., $x \in T$ and $y \in T$. By the lemma from part (a), we can conclude that $x + y \in T$, so $P(x + y)$ holds.

Therefore, $P(x)$ is true for all $x \in S$ by induction.

4. Conceptual Review

(a) Regular expression rules:
Basis: ε, a for $a \in \Sigma$
Recursive: If A, B are regular expressions, $(A \cup B), AB$, and A^* are regular expressions.

5. Regular Expressions Warmup

(a) Consider the following Regular Expression (RegEx):

$$1(45 \cup 54)^{\star}1$$

List 5 strings that are accepted by the RegEx and 5 strings that are rejected. The strings should be over the alphabet $\Sigma := \{1, 4, 5\}$. After listing the strings, summarize the RegEx in your own words.

Solution:

Accepted:

- 1451
- 1541
- 145541
- 1454545451
- 11

Rejected:

- 1
- 1441
- 45
- 14451
- 111

This RegEx accepts exactly those strings that start with a 1 and end with a 1, and have zero or more copies of 45 or 54 in the middle.

(b) Consider the following Regular Expression (RegEx):

$$a(aaa)^*(bb)^*$$

List 5 strings that are accepted by the RegEx and 5 strings that are rejected. The strings should be over the alphabet $\Sigma := \{a, b\}$. After listing the strings, summarize the RegEx in your own words.

Solution:

Accepted:

- a
- aaaa
- abb
- abbbb
- aaaaaaabbbbbb

Rejected:

- ε
- aa
- aaa
- ab
- abba

This RegEx accepts exactly those strings that start with a run of 'a's with length equivalent to 1 mod 3 followed by an even number of 'b's.

6. Constructing RegExes

For each of the following, construct a regular expression for the specified language.

(a) Strings over the alphabet $\Sigma := \{a, b\}$ with odd length.

Solution:

RegEx:

$$(aa \cup ab \cup ba \cup bb)^*(a \cup b)$$

(b) Strings over the alphabet $\Sigma := \{a\}$ with an even number of a's.

Solution:

RegEx:

$$(aa)^*$$

(c) Strings over the alphabet $\Sigma := \{a, b\}$ with an even number of a's.

Solution:

RegEx:

$$b^*(b^*ab^*ab^*)^*$$

Alternate solution:

$$b^*(ab^*ab^*)^*$$

The extra b^* isn't necessary because we can add any number of b's we want before the very first a using the b^* before the parentheses. Then, we can use the b^* after the second a to add as many b's as we want before the next pair of a's.

Another solution!

$$(b \cup (ab^*a))^*$$

(d) Strings over the alphabet $\Sigma := \{a, b\}$ with alternating a's and b's (i.e., not containing aa or bb).

Solution:

RegEx:

$$(a \cup \varepsilon)(ba)^*(b \cup \varepsilon)$$

Alternate solution:

$$(b \cup \varepsilon)(ab)^*(a \cup \varepsilon)$$

(e) Strings over the alphabet $\Sigma := \{a, b\}$ where the second to last character is a b.

Solution:

RegEx:

$$(a \cup b)^*(bb \cup ba)$$

Alternate solution:

$$(a \cup b)^*b(b \cup a)$$

(f) Strings over the alphabet $\Sigma := \{a, b\}$ not ending in aa.

Solution:

RegEx:

$$\varepsilon \cup a \cup b \cup ((a \cup b)^*(bb \cup ab \cup ba))$$

7. Powerful Ideas

Let A and B be sets. Consider the following claim:

If $A \subseteq B$ then $\mathcal{P}(A) \subseteq P(B)$

Write an **English proof** that the claim holds.

Solution:

Let X be an arbitrary element of $\mathcal{P}(A)$. By definition of power set, $X \subseteq A$. Let x be an arbitrary element of X. Since Since $X \subseteq A$, by definition of subset, $x \in A$. Since $A \subseteq B$, by definition of subset, $x \in B$. Since x was an arbitrary element of X, by definition of subset, $X \subseteq B$. By definition of power set, $X \in \mathcal{P}(B)$. Since X was an arbitrary element of $\mathcal{P}(A)$, by definition of subset, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

8. Set Equality Proof II

Let A, B, C be sets. Consider the following claim

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$$

(a) Write a **formal proof** that the claim holds.

Solution:

This solution refers to 'Def' when applying a definition "forwards" (replacing set notation with predicate logic), and 'undef' when applying a definition "backwards" (replacing predicate logic with set notation). In a 'real' 311 submission, you'd call this a 'definition' in both spots, but the different names here might help you see the pattern a bit more easily.

Let x be arbitrary

1.1.1	$x \in A \setminus (B \cap C)$	Assumption
1.1.2	$x \in A \wedge \neg(x \in B \cap C)$	Def of Set Difference 1.1.1
1.1.3	$x \in A \wedge \neg(x \in B \wedge x \in C)$	Def of Intersection 1.1.2
1.1.4	$x \in A \wedge (\neg(x \in B) \vee \neg(x \in C))$	De Morgan 1.1.3
1.1.5	$(x \in A \wedge \neg(x \in B)) \vee (x \in A \wedge \neg(x \in C))$	Distributivity 1.1.4
1.1.6	$(x \in A \setminus B) \vee (x \in A \wedge \neg(x \in C))$	Undef Set Difference 1.1.5
1.1.7	$(x \in A \setminus B) \vee (x \in A \setminus C)$	Undef Set Difference 1.1.6
1.1.8	$x \in (A \setminus B) \cup (A \setminus C)$	Undef Union 1.2.7
1.1	$x \in A \setminus (B \cap C) \rightarrow x \in (A \setminus B) \cup (A \setminus C)$	Direct Proof
1.2.1	$x \in (A \setminus B) \cup (A \setminus C)$	Assumption
1.2.2	$(x \in A \setminus B) \vee (x \in A \setminus C)$	Def of Union 1.2.1
1.2.3	$(x \in A \wedge \neg(x \in B)) \vee (x \in A \setminus C)$	Def of Set Difference 1.2.3
1.2.4	$(x \in A \wedge \neg(x \in B)) \vee (x \in A \wedge \neg(x \in C))$	Def of Set Difference 1.2.3
1.2.5	$x \in A \wedge (\neg(x \in B) \vee \neg(x \in C))$	Distributivity 1.2.4
1.2.6	$x \in A \wedge \neg(x \in B \wedge x \in C)$	De Morgan 1.2.5
1.2.7	$x \in A \wedge \neg(x \in B \cap C)$	Undef Intersection 1.2.6
1.2.8	$x \in A \setminus (B \cap C)$	Undef Set Difference 1.1.7
1.2	$x \in (A \setminus B) \cup (A \setminus C) \rightarrow x \in A \setminus (B \cap C)$	Direct Proof
1.3	$(x \in A \setminus (B \cap C) \rightarrow x \in (A \setminus B) \cup (A \setminus C)) \wedge (x \in (A \setminus B) \cup (A \setminus C) \rightarrow x \in A \setminus (B \cap C))$	

Intro $\wedge$ 1.1, 1.2

1.4 $x \in A \setminus (B \cap C) \leftrightarrow x \in (A \setminus B) \cup (A \setminus C)$ — Biconditional 1.2, 1.3

1. $\forall x, x \in A \setminus (B \cap C) \leftrightarrow x \in (A \setminus B) \cup (A \setminus C)$ — Intro $\forall$
2. $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ — Undef SameSet 1

(b) Translate your proof to an **English Proof**.
Follow the Meta-Theorem template from lecture (i.e., using a chain of equivalences instead of two subset proofs).

Solution:

Let x be arbitrary. We show being an element of the left set and being an element of the right set are equivalent:

$$
\begin{aligned}
x \in A \setminus (B \cap C) &\equiv (x \in A) \wedge \neg(x \in B \cap C) && \text{Def of Set Difference} \\
&\equiv (x \in A) \wedge \neg((x \in B) \wedge (x \in C)) && \text{Def of Intersection} \\
&\equiv (x \in A) \wedge (\neg(x \in B) \vee \neg(x \in C)) && \text{DeMorgan's Law} \\
&\equiv ((x \in A) \wedge \neg(x \in B)) \vee ((x \in A) \wedge \neg(x \in C)) && \text{Distributivity} \\
&\equiv (x \in A \setminus B) \vee (x \in A \setminus C) && \text{Def of Set Difference} \\
&\equiv x \in (A \setminus B) \cup (A \setminus C) && \text{Def of Union}
\end{aligned}
$$

Since x was arbitrary, we have shown $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

(c) Optional: Re-write this proof as an **English Proof** that is made up of two subset proofs.

Solution:

Let $x \in A \setminus (B \cap C)$ be arbitrary. Then by definition of set difference, $x \in A$ and $x \notin B \cap C$. Then by definition of intersection and DeMorgan's Law, $x \notin B$ or $x \notin C$. Thus (by distributive property of propositions) we have $x \in A$ and $x \notin B$, or $x \in A$ and $x \notin C$. Then by definition of set difference, $x \in (A \setminus B)$ or $x \in (A \setminus C)$. Then by definition of union, $x \in (A \setminus B) \cup (A \setminus C)$. Since x was arbitrary, we have shown $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$.

Let $x \in (A \setminus B) \cup (A \setminus C)$ be arbitrary. Then by definition of union, $x \in (A \setminus B)$ or $x \in (A \setminus C)$. Then by definition of set difference, $x \in A$ and $x \notin B$, or $x \in A$ and $x \notin C$. Then (by distributive property of propositions) $x \in A$, and $x \notin B$ or $x \notin C$. Then by definition of intersection and DeMorgan's Law, $x \in A$ and $x \notin (B \cap C)$. Then by definition of set difference, $x \in A \setminus (B \cap C)$. Since x was arbitrary, we have shown that $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$.

Since $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$ and $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$, we have shown $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.