CSE 390Z: Mathematics for Computation Workshop

Week 6 Workshop Solutions

Conceptual Review

(a) **Set Operations and Comparisons**

Set Equality: $A = B := \forall x(x \in A \leftrightarrow x \in B)$
Subset: $A \subseteq B := \forall x(x \in A \rightarrow x \in B)$
Union: $A \cup B := \{x \ : \ x \in A \lor x \in B\}$
Intersection: $A \cap B := \{x \ : \ x \in A \land x \in B\}$
Set Difference: $A \setminus B = A - B := \{x \ : \ x \in A \land x \notin B\}$
Set Complement: $\overline{A} = A^C := \{x \ : \ x \notin A\}$
Powerset: $\mathcal{P}(A) := \{B \ : B \subseteq A\}$
Cartesian Product: $A \times B := \{(a, b) \ : a \in A, \ b \in B\}$

(b) **Set Builder Notation**

Filter: $S := \{x \in U : P(x)\}$
Translation: S is all the things in U that satisfy $P(x)$.

Map: $T := \{f(x) : x \in U\}$
Translation: T is all output values from the function $f(x)$ when the input is something from U.

The : is read as "such that". It is also common to use | instead of :. When using set builder notation, the stuff before the : (or |) is the stuff in the set. The stuff after the : (or |) are requirements that stuff must fulfill to be in the set.

(c) How do we prove that for sets A and B, $A \subseteq B$?

Solution:

Let $x \in A$ be arbitrary... thus $x \in B$. Since x was arbitrary, we have proven, by the definition of subset, that $A \subseteq B$.

(d) What are two ways we can prove that for sets A and B, $A = B$?

Solution:

Use two subset proofs to show that $A \subseteq B$ and $B \subseteq A$. OR
Using a chain of equivalences (This is the preferred method when A and B are defined in terms of set operations):
Let x be an arbitrary $<<$thing in the domain$>>$
The stated biconditional holds since

$$\begin{aligned} x \in A &\equiv << \text{ replace set operations with logical operators } >> \\ &\equiv << \text{ apply propositional logic equivalences } >> \\ &\equiv << \text{ replace logical operators with set operations } >> \\ &\equiv x \in B \end{aligned}$$

Since x was arbitrary, we have proven, by the definition of set equality, that $A = B$.

1. A Basic Subset Proof

Let A, B be sets. Consider the following claim:

$$A \cap B \subseteq A \cup B$$

(a) Write a **formal proof** that the claim holds. Use cozy-style rules for applying definitions. For example, You can replace $A \subseteq B$ by $\forall x(x \in A \to x \in B)$ with "Def of Subset" and the reverse with "Undef Subset".

Solution:

Let x be arbitrary	
1.1.1 $x \in A \cap B$	Assumption
1.1.2 $x \in A \wedge x \in B$	Def of Intersection: 1.1.1
1.1.3 $x \in A$	Elim $\wedge$: 1.1.2
1.1.4 $x \in A \vee x \in B$	Intro $\vee$ 1.1.3
1.1.5 $x \in A \cup B$	Undef Union 1.1.4
1.1 $x \in A \cap B \to x \in A \cup B$	Direct Proof
1. $\forall x, x \in A \cap B \to x \in A \cup B$	Intro $\forall$
2. $A \cap B \subseteq A \cup B$	Undef Subset: 1

(b) Translate your formal proof to an **English proof**. You may be surprised by how short your proof is!

Solution:

Let $x \in A \cap B$ be arbitrary. Then by definition of intersection, $x \in A$ and $x \in B$. Since $x \in A$, we have $x \in A$ or $x \in B$. Then by definition of union, $x \in A \cup B$. Since x was arbitrary, this shows that $A \cap B \subseteq A \cup B$.

2. Set Equality Proof

(a) Write an English proof to show that $A \cap (A \cup B) \subseteq A$ for sets A, B.

Solution:

Let x be an arbitrary member of $A \cap (A \cup B)$. Then by definition of intersection, $x \in A$ and $x \in A \cup B$. So certainly, $x \in A$. Since x was arbitrary, we have shown that $A \cap (A \cup B) \subseteq A$ by definition of subset.

(b) Write an English proof to show that $A \subseteq A \cap (A \cup B)$ for sets A, B.

Solution:

Let $y \in A$ be arbitrary. Since $y \in A$, we have $y \in A$ or $y \in B$. Then by definition of union, $y \in A \cup B$. Since $y \in A$ and $y \in A \cup B$, by definition of intersection, $y \in A \cap (A \cup B)$. Since y was arbitrary, we have shown that $A \subseteq A \cap (A \cup B)$.

(c) Combine part (a) and (b) to conclude that $A \cap (A \cup B) = A$ for sets A, B.

Solution:

Since $A \cap (A \cup B) \subseteq A$ and $A \subseteq A \cap (A \cup B)$, we have shown that $A \cap (A \cup B) = A$.

(d) Re-write this proof using the Meta-Theorem template from lecture (i.e., using a chain of equivalences instead of two subset proofs).

Solution:

Let x be arbitrary. The biconditional $\forall x(x \in A \cap (A \cup B) \leftrightarrow x \in A)$ holds since

$$\begin{aligned} x \in A \cap (A \cup B) &\equiv (x \in A) \wedge (x \in A \cup B) && \text{Def of Intersection} \\ &\equiv (x \in A) \wedge (x \in A \vee x \in B) && \text{Def of Union} \\ &\equiv x \in A && \text{Absorption} \end{aligned}$$

Since x was arbitrary, we have proven, by definition of set equality, that $A \cap (A \cup B) = A$.

3. Subsets

Let A, B, C be sets. Consider the following claim:

$A \subseteq C$ follows from $A \subseteq B$ and $B \subseteq C$

(a) Write a **formal proof** that the claim holds:

Solution:

1. $A \subseteq B$ — Given
2. $B \subseteq C$ — Given
3. $\forall x, x \in A \to x \in B$ — Def of Subset: 1
4. $\forall x, x \in B \to x \in C$ — Def of Subset: 2

   Let x be arbitrary.

   5.1.1 $x \in A$ — Assumption

   5.1.2 $x \in A \to x \in B$ — Elim $\forall$: 3

   5.1.3 $x \in B$ — Modus Ponens: 5.1.1, 5.1.2

   5.1.4 $x \in B \to x \in C$ — Elim $\forall$: 4

   5.1.5 $x \in C$ — Modus Ponens: 5.1.3, 5.1.4

   5.1 $x \in A \to x \in C$ — Direct Proof

5. $\forall x, x \in A \to x \in C$ — Intro $\forall$
6. $A \subseteq C$ — Undef Subset: 5

(b) Translate the formal proof to an **English Proof**.

Solution:

Let x be an arbitrary element of A. Since $A \subseteq B$, by definition of subset, $x \in B$. Then, since $B \subseteq C$, by definition of subset, $x \in C$. Since x was arbitrary, we have shown that $A \subseteq C$ by definition of subset.

4. Moderately Unsettling

Let A, B and C be the following sets:

$$A := \{x \in \mathbb{Z} : x \equiv_4 0\}$$
$$B := \{x \in \mathbb{Z} : x \equiv_4 2\}$$
$$C := \{x \in \mathbb{Z} : x \equiv_2 0\}$$

Consider the following claim:

$$C = (A \cup B)$$

(a) Write an English proof to show that $C \subseteq (A \cup B)$

Solution:

Let x be an arbitrary element of C. By definition of C, we have $x \equiv_2 0$. By definition of congruence, $2|x$ and by definition of divides, $x = 2k$ for some integer k. We proceed by cases:

Case 1: Suppose k is even. By definition of even, $k = 2m$ for some integer m. Then $x = 2k = 2(2m) = 4m$. By definition of divides, $4|x$ and by definition of congruence $x \equiv_4 0$. By definition of A, $x \in A$. Since $x \in A$, $x \in A$ or $x \in B$, and by definition of union, $x \in (A \cup B)$

Case 2: Suppose k is odd. By definition of odd, $k = 2n + 1$ for some integer n. Then $x = 2k = 2(2n+1) = 4n+2$. By definition of divides, $4|x-2$ and by definition of congruence $x \equiv_4 2$. By definition of B, $x \in B$. Since $x \in B$, $x \in A$ or $x \in B$, and by definition of union, $x \in (A \cup B)$

Since these cases are exhaustive, we have shown that $x \in (A \cup B)$.
Since x was arbitrary, we have shown that $C \subseteq (A \cup B)$.

(b) Write an English proof to show that $(A \cup B) \subseteq C$

Solution:

Let x be an arbitrary element of $A \cup B$. By definition of union, $x \in A$ or $x \in B$. We proceed by cases:

Case 1: Suppose $x \in A$. By definition of A, $x \equiv_4 0$. By definition of congruence, $4|x$, and by definition of divides, $x = 4k = 2(2k)$ for some integer k. By definition of divides, $2|x$, and by definition of congruence $x \equiv_2 0$. By definition of C, $x \in C$.

Case 2: Suppose $x \in B$. By definition of B, $x \equiv_4 2$. By definition of congruence $4|(x - 2)$, and by definition of divides, $x - 2 = 4j$ for some integer j. Rearranging, we have $x = 4j + 2 = 2(2j + 1)$. By definition of divides, $2|x$, and by definition of congruence, $x \equiv_2 0$. By definition of C, $x \in C$.

Since these cases are exhaustive, we have shown that $x \in C$.
Since x was arbitrary, we have shown that $(A \cup B) \subseteq C$.

(c) Combine part(a) and part(b) to show that $C = (A \cup B)$

Solution:

Since $C \subseteq (A \cup B)$ and $(A \cup B) \subseteq C$, we have shown that $C = (A \cup B)$.

5. $\cup \to \cap$?

Prove or disprove: for all sets A and B, $A \cup B \subseteq A \cap B$.
Recall that we can disprove a for all claim by finding a counter-example.

Solution:

We disprove the claim with a counter example. Consider the sets $A = \{1, 2\}$ and $B = \{1, 3\}$. $A \cup B = \{1, 2, 3\}$ and $A \cap B = \{1\}$. Since $A \cup B$ has elements that are not in $A \cap B$ (2 and 3), by definition of subset, $A \cup B \not\subseteq A \cap B$.

6. Powerful Ideas

Let A and B be sets. Consider the following claim:

If $A \subseteq B$ then $\mathcal{P}(A) \subseteq P(B)$

Write an **English proof** that the claim holds.

Solution:

Let X be an arbitrary element of $\mathcal{P}(A)$. By definition of power set, $X \subseteq A$. Let x be an arbitrary element of X. Since Since $X \subseteq A$, by definition of subset, $x \in A$. Since $A \subseteq B$, by definition of subset, $x \in B$. Since x was an arbitrary element of X, by definition of subset, $X \subseteq B$. By definition of power set, $X \in \mathcal{P}(B)$. Since X was an arbitrary element of $\mathcal{P}(A)$, by definition of subset, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

7. Cartesian Product Proof

Let A, B, C, D be sets. Write an **English proof** of the follow claim:

$$A \times C \subseteq (A \cup B) \times (C \cup D)$$

Solution:

Let $x \in A \times C$ be arbitrary. Then x is of the form $x = (y, z)$, where $y \in A$ and $z \in C$. Since $y \in A$ we have $y \in A$ or $y \in B$. Then by definition of union, $y \in (A \cup B)$. Similarly, since $z \in C$, we have $z \in C$ or $z \in D$. Then by definition of union, $z \in (C \cup D)$. Since $y \in (A \cup B)$ and $z \in (C \cup D)$, we have shown that $x = (y, z) \in (A \cup B) \times (C \cup D)$. Since x was arbitrary, we have shown $A \times C \subseteq (A \cup B) \times (C \cup D)$.

8. Set Equality Proof II

Let A, B, C be sets. Consider the following claim

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$$

(a) Write a **formal proof** that the claim holds.

Solution:

Let x be arbitrary

1.1.1	$x \in A \setminus (B \cap C)$	Assumption
1.1.2	$x \in A \wedge \neg(x \in B \cap C)$	Def of Set Difference 1.1.1
1.1.3	$x \in A \wedge \neg(x \in B \wedge x \in C)$	Def of Intersection 1.1.2
1.1.4	$x \in A \wedge (\neg(x \in B) \vee \neg(x \in C))$	De Morgan 1.1.3
1.1.5	$(x \in A \wedge \neg(x \in B)) \vee (x \in A \wedge \neg(x \in C))$	Distributivity 1.1.4
1.1.6	$(x \in A \setminus B) \vee (x \in A \wedge \neg(x \in C))$	Undef Set Difference 1.1.5
1.1.7	$(x \in A \setminus B) \vee (x \in A \setminus C)$	Undef Set Difference 1.1.6
1.1.8	$x \in (A \setminus B) \cup (A \setminus C)$	Undef Union 1.2.7
1.1	$x \in A \setminus (B \cap C) \rightarrow x \in (A \setminus B) \cup (A \setminus C)$	Direct Proof
1.2.1	$x \in (A \setminus B) \cup (A \setminus C)$	Assumption
1.2.2	$(x \in A \setminus B) \vee (x \in A \setminus C)$	Def of Union 1.2.1
1.2.3	$(x \in A \wedge \neg(x \in B)) \vee (x \in A \setminus C)$	Def of Set Difference 1.2.3
1.2.4	$(x \in A \wedge \neg(x \in B)) \vee (x \in A \wedge \neg(x \in C))$	Def of Set Difference 1.2.3
1.2.5	$x \in A \wedge (\neg(x \in B) \vee \neg(x \in C))$	Distributivity 1.2.4

1.2.6 $x \in A \land \neg(x \in B \land x \in C)$ De Morgan 1.2.5

1.2.7 $x \in A \land \neg(x \in B \cap C)$ Undef Intersection 1.2.6

1.2.8 $x \in A \setminus (B \cap C)$ Undef Set Difference 1.1.7

1.2 $x \in (A \setminus B) \cup (A \setminus C) \to x \in A \setminus (B \cap C)$ Direct Proof

1.3 $(x \in A \setminus (B \cap C) \to x \in (A \setminus B) \cup (A \setminus C)) \land (x \in (A \setminus B) \cup (A \setminus C) \to x \in A \setminus (B \cap C))$ Intro $\land$ 1.1, 1.2

1.4 $x \in A \setminus (B \cap C) \leftrightarrow x \in (A \setminus B) \cup (A \setminus C)$ Biconditional 1.2, 1.3

1. $\forall x, x \in A \setminus (B \cap C) \leftrightarrow x \in (A \setminus B) \cup (A \setminus C)$ Intro $\forall$
2. $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ Undef SameSet 1

(b) Translate your proof to an **English Proof**.
Follow the Meta-Theorem template from lecture (i.e., using a chain of equivalences instead of two subset proofs).

Solution:

Let x be arbitrary. We show being an element of the left set and being an element of the right set are equivalent:

$x \in A \setminus (B \cap C)$	$\equiv (x \in A) \land \neg(x \in B \cap C)$	Def of Set Difference
	$\equiv (x \in A) \land \neg((x \in B) \land (x \in C))$	Def of Intersection
	$\equiv (x \in A) \land (\neg(x \in B) \lor \neg(x \in C))$	DeMorgan's Law
	$\equiv ((x \in A) \land \neg(x \in B)) \lor ((x \in A) \land \neg(x \in C))$	Distributivity
	$\equiv (x \in A \setminus B) \lor (x \in A \setminus C)$	Def of Set Difference
	$\equiv x \in (A \setminus B) \cup (A \setminus C)$	Def of Union

Since x was arbitrary, we have shown $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

(c) Optional: Re-write this proof as an **English Proof** that is made up of two subset proofs.

Solution:

Let $x \in A \setminus (B \cap C)$ be arbitrary. Then by definition of set difference, $x \in A$ and $x \notin B \cap C$. Then by definition of intersection and DeMorgan's Law, $x \notin B$ or $x \notin C$. Thus (by distributive property of propositions) we have $x \in A$ and $x \notin B$, or $x \in A$ and $x \notin C$. Then by definition of set difference, $x \in (A \setminus B)$ or $x \in (A \setminus C)$. Then by definition of union, $x \in (A \setminus B) \cup (A \setminus C)$. Since x was arbitrary, we have shown $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$.

Let $x \in (A \setminus B) \cup (A \setminus C)$ be arbitrary. Then by definition of union, $x \in (A \setminus B)$ or $x \in (A \setminus C)$. Then by definition of set difference, $x \in A$ and $x \notin B$, or $x \in A$ and $x \notin C$. Then (by distributive property of propositions) $x \in A$, and $x \notin B$ or $x \notin C$. Then by definition of intersection and DeMorgan's Law, $x \in A$ and $x \notin (B \cap C)$. Then by definition of set difference, $x \in A \setminus (B \cap C)$. Since x was arbitrary, we have shown that $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$.

Since $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$ and $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$, we have shown $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

9. Strong Induction: Packs of Candy

A store sells candy in packs of 4 and packs of 7. Let P(n) be defined as "You are able to buy n packs of candy". For example, $P(3)$ is not true, because you cannot buy exactly 3 packs of candy from the store. However, it turns out that P(n) is true for all $n \geq 18$. Use strong induction to prove this.

Hint: It may be easier to leave your base cases blank, write your inductive step, then figure out how many base cases you need, and go back and fill them in.

Solution:

Let P(n) be defined as "You are able to buy n packs of candy". We will prove $P(n)$ is true for all integers $n \geq 18$ by strong induction.

Base Cases: $(n = 18, 19, 20, 21)$**:**

- $n = 18$: 18 packs of candy can be made up of 2 packs of 7 and 1 pack of 4 $(18 = 2 * 7 + 1 * 4)$.
- $n = 19$: 19 packs of candy can be made up of 1 pack of 7 and 3 packs of 4 $(19 = 1 * 7 + 3 * 4)$.
- $n = 20$: 20 packs of candy can be made up of 5 packs of 4 $(20 = 0 * 7 + 5 * 4)$.
- $n = 21$: 21 packs of candy can be made up of 3 packs of 7 $(21 = 3 * 7 + 0 * 4)$.

Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 21$, we can buy j packs of candy for every integer j from 18 to k.
In other words, assume P(18) ∧... ∧P(k) hold for some arbitrary integer $k \geq 21$.

Inductive Step:

Goal: Show $P(k + 1)$, i.e. show that we can buy $k + 1$ packs of candy.

We want to buy $k + 1$ packs of candy. Since $k \geq 21$, $(k + 1) - 4 = k - 3 \geq 18$.
Our inductive hypothesis covers everything from 18 to k and $18 \leq k - 3 \leq k$
Therefore $k - 3$ is covered by our inductive hypothesis.
So, by the I.H., we can buy exactly $k - 3$ packs of candy. We can add another 4 packs in order to buy $k + 1$ packs of candy, so P$(k + 1)$ is true.

Conclusion: By strong induction, P(n) is true for all integers $n \geq 18$.

Note: Notice that we use the fact that $k - 3$ was covered by our inductive hypothesis as part of our proof. Since 18 is the smallest value in our domain, we need $k - 3 \geq 18$. Adding 3 to both sides this means $k \geq 21$. That's how we knew we needed the largest value in our base case to be 21.
Some people find it helpful to think of it this way: we had to use a fact from 4 steps back from $k + 1$ to $k - 3$ in the IS, so we needed 4 base cases.