CSE 390Z: Mathematics for Computation Workshop

Mid-Quarter Review Solutions

Name: ______________________________

1. Predicate Translation

Let the domain of discourse be novels, comic books, movies, and TV shows. Translate the following statements to predicate logic, using the following predicates:

Novel(x) := x is a novel
Comic(x) := x is a comic book
Movie(x) := x is a movie
Show(x) := x is a TV show
Adaptation(x, y) := x is an adaptation of y
You may use $=$ as a predicate to test if two things are the same.

(a) Every movie is an adaptation of a novel or a comic book.

Solution:

$\forall m(\text{Movie}(m) \rightarrow \exists x[(\text{Novel}(x) \vee \text{Comic}(x)) \wedge \text{Adaptation}(m, x)])$

(b) Every movie is an adaptation of exactly one novel.

Solution:

$\forall m(\text{Movie}(m) \rightarrow \exists x(\text{Novel}(x) \wedge \text{Adaptation}(m, x) \wedge \forall y[(\text{Novel}(y) \wedge \text{Adaptation}(m, y)) \rightarrow (y = x)]))$

OR

$\forall m(\text{Movie}(m) \rightarrow \exists x(\text{Novel}(x) \wedge \text{Adaptation}(m, x) \wedge \forall y[(y \neq x) \rightarrow \neg(\text{Novel}(y) \wedge \text{Adaptation}(m, y))]))$

OR

$\forall m(\text{Movie}(m) \rightarrow \exists x \forall y[(y = x) \leftrightarrow (\text{Novel}(y) \wedge \text{Adaptation}(m, y))])$

(c) Using the same domain of discourse and predicates as above, translate the following predicate logic statement into English. Your translation should be as natural as possible.

$\forall x(\text{Novel}(x) \rightarrow (\neg\exists m[\text{Adaptation}(m, x) \wedge \text{Movie}(m)] \vee \neg\exists s[\text{Adaptation}(s, x) \wedge \text{Show}(s)]))$

Solution:

A novel cannot be adapted into both a movie and a TV show.

2. Normal Forms

Consider the following function F:

p	q	r	$F(p,q,r)$
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	T

(a) Write a propositional logic expression for F in DNF form (ORs of ANDs).

Solution:

$$(p \land \neg q \land r) \lor (\neg p \land q \land r) \lor (\neg p \land \neg q \land \neg r)$$

(b) Write a propositional logic expression for F in CNF form (ANDs of ORs).

Solution:

$$(\neg p \lor \neg q \lor \neg r) \land (\neg p \lor \neg q \lor r) \land (\neg p \lor q \lor r) \land (p \lor \neg q \lor r) \land (p \lor q \lor \neg r)$$

3. Modular Arithmetic

Write an **English proof** that for all integers $x, y, n > 0$, if $x \equiv_{6n} 1$ and $y \equiv_{7n} 5$ then $7x + 2y \equiv_{14n} 17$.

Hint: Apply the definition of congruence and divides.

Solution:

Let $x, y, n > 0$ be arbitrary integers. Suppose $x \equiv_{6n} 1$ and $y \equiv_{7n} 5$. Then by definition of congruence, $6n \mid (x-1)$ and $7n \mid (y-5)$. Then by definition of divides, there exists integers j, k such that $x - 1 = 6nk$ and $y - 5 = 7nj$. Thus $x = 6nk + 1$ and $y = 7nj + 5$. Then we have:

$$\begin{aligned} 7x + 2y &= 7(6nk+1) + 2(7nj+5) \\ &= 42nk + 7 + 14nj + 10 \\ &= 42nk + 14nj + 17 \\ &= 14n(3k+j) + 17 \end{aligned}$$

We have shown that $(7x + 2y) - 17 = 14n(3k + j)$. So by definition of divides, $14n \mid (7x + 2y) - 17$. Then by definition of congruence, $7x + 2y \equiv_{14n} 17$. Since x, y, n were arbitrary, the claim holds.

4. Extended Euclidean Algorithm

Solve the equation and state the full set of solutions

$$31y \equiv_{83} 2$$

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Solution:

First we use the Euclidean algorithm to compute gcd(83,31) keeping track of our equations in every step

$\gcd(83, 31) = \gcd(31, 83\%31) = \gcd(31, 21)$	$83 = 2 * 31 + 21$
$\gcd(31, 21) = \gcd(21, 31\%21) = \gcd(21, 10)$	$31 = 1 * 21 + 10$
$\gcd(21, 10) = \gcd(10, 21\%10) = \gcd(10, 1)$	$21 = 2 * 10 + 1$
$\gcd(10, 1) = \gcd(1, 10\%1) = \gcd(1, 0) = 1$	no equation for this line

Now, we rearrange:

$$83 = 2 * 31 + 21 \qquad 21 = 83 - 2 * 31 \tag{1}$$

$$31 = 1 * 21 + 10 \qquad 10 = 31 - 1 * 21 \tag{2}$$

$$21 = 2 * 10 + 1 \qquad 1 = 21 - 2 * 10 \tag{3}$$

Now we use back substitution to find an equation of the form $\gcd(83, 31) = s * 83 + t * 31$.
The labels used below are from the previous step.

$$\begin{aligned}
1 &= 21 - 2 * 10 && \text{Start with equation (3)} \\
&= 21 - 2 * (31 - 1 * 21) && \text{Sub in equation (2)} \\
&= 21 - 2 * 31 + 2 * 21 \\
&= -2 * 31 + 3 * 21 \\
&= -2 * 31 + 3 * (83 - 2 * 31) && \text{Sub in equation (1)} \\
&= -2 * 31 + 3 * 83 - 6 * 31 \\
&= 3 * 83 - 8 * 31
\end{aligned}$$

So -8 is our multiplicative inverse. Since we want an inverse j such that $0 \leq j < 83$ we use $-8 + 83 = 75$ as our multiplicative inverse.
We multiply both sides of our original equation $31y \equiv_{83} 2$ by 75.

$$\begin{aligned}
75 \cdot 31y &\equiv_{83} 75 \cdot 2 \\
y &\equiv_{83} 150 \equiv_{83} 67
\end{aligned}$$

So the full set of solutions is $67 + 83k$ for any integer k.

5. Induction

Prove by induction that $3^n - 1$ is divisible by 2 for any integer $n \geq 1$.

Solution:

1. Let P(n) be the statement "$3^n - 1$ is divisible by 2". We prove P(n) for all integers $n \geq 1$ by induction.

2. Base Case ($n = 1$): $3^n - 1 = 3^1 - 1 = 3 - 1 = 2 = 2 \cdot 1$. By definition of divides, $2 \mid 2$. So $P(1)$ holds.

3. Inductive Hypothesis:
Suppose that P(k) holds for some arbitrary integer $k \geq 1$. Then $2 \mid 3^k - 1$. Then by definition of divides, there exists some integer a such that $3^k - 1 = 2a$.

4. Inductive Step: Goal: Prove $P(k+1)$ i.e., prove that $2|(3^{k+1} - 1)$.

We have:

$$
\begin{aligned}
3^{k+1} - 1 &= 3(3^k) - 1 \\
&= 3(3^k - 1 + 1) - 1 && \text{Creatively add 0 (0 = -1+1)} \\
&= 3(2a + 1) - 1 && \text{By IH} \\
&= 6a + 3 - 1 \\
&= 6a + 2 \\
&= 2(3a + 1)
\end{aligned}
$$

Thus by definition of divides, $2 \mid 3^{k+1} - 1$. So P($k + 1$) holds.

5. We have shown that P(n) holds for all integers $n \geq 1$ by induction.

6. Strong Induction

Consider the function f, which takes a natural number as input and outputs a natural number.

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ f(n-1) + 2 \cdot f(n-2) & \text{if } n \geq 2 \end{cases}$$

Prove that $f(n) = 2^n$ for all $n \in \mathbb{N}$.

Solution:

1. Let $P(n)$ be the claim that $f(n) = 2^n$. We will prove $P(n)$ holds for all $n \in \mathbb{N}$ by strong induction.

2. Base Cases (n = 0, n = 1):
$f(0) = 1 = 1 = 2^0$ so $P(0)$ holds.
$f(1) = 2 = 2 = 2^1$ so $P(1)$ holds.

3. Inductive Hypothesis:
Option 1: Suppose that $P(j)$ holds for all $j \in \mathbb{N}$ such that $j \leq k$ for some arbitrary integer $k \geq 1$.
Option 2: Suppose that $P(0) \wedge P(1) \wedge ... \wedge P(k)$ for some arbitrary integer $k \geq 1$.

4. Inductive Step: Goal: Show $P(k+1)$ i.e., show that $f(k+1) = 2^{k+1}$

Since $k \geq 1$, $k+1 \geq 2$, so we are guaranteed to be in the recursive part of $f(n)$. We have:

$$\begin{aligned} f(k+1) &= f(k+1-1) + 2 \cdot f(k+1-2) && \text{Definition of } f \\ &= f(k) + 2 \cdot f(k-1) \\ &= 2^k + 2 \cdot 2^{k-1} && \text{I.H. twice} \\ &= 2^k + 2^{k-1+1} \\ &= 2^k + 2^k \\ &= 2 \cdot 2^k \\ &= 2^{k+1} \end{aligned}$$

So, $P(k+1)$ holds.

5. Therefore, we have proven that the claim holds for all $n \in \mathbb{N}$ by strong induction.