CSE 390Z: Mathematics for Computation Workshop

Practice 311 Final

Name: ______________________________

UW ID: ______________________________

Instructions:

- This is a **simulated practice final**. You will **not** be graded on your performance on this exam.
- Nevertheless, please treat this as if it is a real exam. That means that you may not discuss with your neighbors, reference outside material, or use your devices during the next 100 minute period.
- There are 9 "graded" problems on this exam and 2 bonus problems (we think the 9 “graded” problems would constitute a long (but could fit in 2 hours) 311 final; the bonus problems are helpful practice if you have more time).

1. Predicate Translation

Let the domain of discourse be numbers and mathematicians (math teachers and math students). Let the following predicates be defined:

Number(x) := x is a number
Prime(x) := x is a prime number
MathTeacher(x) := x is a math teacher
MathStudent(x) := x is a math student
Likes(x, y) := x likes y
You can also use $x = y$ and $x \neq y$.

For parts (a) and (b), translate the predicate logic statement into natural English.

(a) $\forall x \forall y((\text{Prime}(x) \land \text{Likes}(y, x)) \rightarrow \text{MathStudent}(y))$

(b) $\exists x(\text{Prime}(x) \land \forall y((\text{MathTeacher}(y) \lor \text{MathStudent}(y)) \rightarrow \text{Likes}(y, x)))$

For parts (c) and (d), translate the English sentence into predicate logic.

(c) There are at least two (different) prime numbers

(d) Math teachers and math students don't like the same numbers.

2. Formal Proof

Write a **formal proof** of the following statement:

For all integers a, b, c, if $a^2 \mid b$ and $b^3 \mid c$, then $a^6 \mid c$.

3. English Proof

Write an **English proof** of the following statement:

For all integers n, $n^2 \equiv_4 0$ or $n^2 \equiv_4 1$

4. Induction I

Prove by induction that $(1+\pi)^n > 1 + n\pi$ for all integers $n \geq 2$.

5. Set Theory

(a) Let A and B be sets. Write an **English proof** of the following claim using the Meta Theorem:

$$A \setminus \overline{B} = (A \cup \varnothing) \cap B$$

Hint: The empty set, $\varnothing$, is the set that contains no elements. In other words, for any x, $x \in \varnothing \equiv F$.

Determine if the following claims are true or false. Explain your reasoning in 1-3 sentences.
You may include images or examples in your explanation. **You do not need to give a proof or disproof.**

(b) For all sets A, B: $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$.

(c) For all sets A, B: $\mathcal{P}(A) \times \mathcal{P}(B) \subseteq \mathcal{P}(A \times B)$.

6. Induction II

Let the set S be recursively defined as follows:
Basis: $(0, 0) \in S$
Recursive Step: If $(x, y) \in S$, then $(x + 2, y + 4) \in S$ and $(x + 4, y + 8) \in S$.

Prove that for all $(x, y) \in S$, $x + y$ is divisible by 3.

7. Relations

For parts (a) and (b), consider the relation $R \subseteq \mathbb{Z} \times \mathbb{Z}$ defined by $(x, y) \in R$ iff $4|(x + y)$

(a) List 3 elements of R.

(b) List the properties that R has out of the following: reflexive, transitive, symmetric, antisymmetric. If R has a property, simply say so without further explanation. If R does not have a property, provide a counterexample without explaining further.

(c) Let R and S be symmetric relations on a set A. Write an **English proof** that $R \setminus S$ is symmetric.

8. Languages

For parts (a), (b), (c) consider the language L containing all binary strings x with the following property:
If there is a 0 at position i in x, then there is a 1 at position $i+2$ in x.
In other words, every time we see a 0, the character after the 0 can be anything, but the character after that has to be a 1.
Some strings in L are: ε, 011, 0011, 011011, 111011. Some strings not in L are: 0, 01, 001, 01011.
Notice that 001 is not in L because even though there is a 1 two characters after the first 0, there is not a 1 two characters after the second 0.

(a) Construct a regular expression that matches the strings in L.

(b) Construct a CFG that generates L.

(c) Construct a DFA that recognizes L.

(d) Let K be the language containing all binary strings x such that both of the following are true:
- x contains at least one 0 **and**
- x ends with 11

Draw an *NFA* that recognizes K.

9. Irregularity/Uncountability

Do exactly **one** of these two problems.

I am doing

◯ part (a) Irregularity

◯ part (b) Uncountability

(a) Prove that the language $L = \{10^x 10^{x+1} 1 : x \geq 0\}$ is not regular.

(b) Olympic champion, Alysa Liu, is so incredible that her next free skate will contain infinitely many jumps. She can do single, double, or triple versions of the following 6 jumps: axel, salchow, loop, flip, toe-loop, and lutz. Prove that the set of free skates Alysa can choose from is uncountable.

10. Bonus Problem: Undecidability

(a) Consider the following problem: Given an arbitrarily large number n and an integer x, output whether or not n has exactly x prime factors.

Is this problem decidable? Provide a brief justification for your answer.

(b) Consider the following problem: Given CODE(P), determine whether $P(x)$ produces the same output as $P(\text{reverse}(x))$ for every input x.

Is this problem decidable? Provide a brief justification for your answer.

11. Bonus Problem: Boolean Algebra

Let f be the boolean function defined as $f(x, y, z) = (x + y)' + (zy)$

(a) Fill in the following table with the values of $f(x, y, z)$ in the last column. Feel free to use the blank columns while doing your work.

x	y	z				$f(x, y, z)$
0	0	0				
0	0	1				
0	1	0				
0	1	1				
1	0	0				
1	0	1				
1	1	0				
1	1	1				

(b) Write $f(x, y, z)$ as a Boolean algebra expression in the canonical sum-of-products (DNF) form.

(c) Write $f(x, y, z)$ as a Boolean algebra expression in the canonical product-of-sums (CNF) form.