

CSE 390Z: Mathematics for Computation Workshop

QuickCheck: 5. Induction Solutions (due Tuesday, May 5th, 9 AM)

Please submit a response to the following questions on Gradescope. We do not grade on accuracy, so please submit your best attempt. You may either typeset your responses or hand-write them. Note that hand-written solutions must be legible to be graded.

We have created [this template](#) if you choose to typeset with Latex. [This guide](#) has specific information about scanning and uploading pdf files to Gradescope.

0. Induction: Equality

For any $n \in \mathbb{N}$, define S_n to be the sum of the squares of the first n positive integers, or

$$S_n = 1^2 + 2^2 + \cdots + n^2.$$

Prove that for all $n \in \mathbb{N}$, $S_n = \frac{1}{6}n(n+1)(2n+1)$.

Note: The empty sum is defined to be zero. This means $S_0 = 0$.

Solution:

Let $P(n)$ be the statement " $S_n = \frac{1}{6}n(n+1)(2n+1)$ " defined for all $n \in \mathbb{N}$. We prove that $P(n)$ is true for all $n \in \mathbb{N}$ by induction on n .

Base Case: When $n = 0$, we know the sum of the squares of the first n positive integers is the sum of no terms, so we have a sum of 0. Thus, $S_0 = 0$. Since $\frac{1}{6}(0)(0+1)((2)(0)+1) = 0$, we know that $P(0)$ is true.

Inductive Hypothesis: Suppose that $P(k)$ is true for some arbitrary $k \in \mathbb{N}$.

Inductive Step: Goal: Show $P(k+1)$, i.e. show $S_{k+1} = \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$

Examining S_{k+1} , we see that

$$S_{k+1} = 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = S_k + (k+1)^2.$$

By the inductive hypothesis, we know that $S_k = \frac{1}{6}k(k+1)(2k+1)$. Therefore, we can substitute and rewrite the expression as follows:

$$\begin{aligned} S_{k+1} &= S_k + (k+1)^2 \\ &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\ &= (k+1) \left(\frac{1}{6}k(2k+1) + (k+1) \right) \\ &= \frac{1}{6}(k+1)(k(2k+1) + 6(k+1)) \\ &= \frac{1}{6}(k+1)(2k^2 + 7k + 6) \\ &= \frac{1}{6}(k+1)(k+2)(2k+3) \\ &= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1) \end{aligned}$$

Thus, we can conclude that $P(k+1)$ is true.

Conclusion: $P(n)$ holds for all integers $n \geq 0$ by the principle of induction.

1. Video Solution

Watch **this video** on the solution **after** making an initial attempt. Then, answer the following questions.

- (a) What is one thing you took away from the video solution?