CSE 390Z: Mathematics for Computation Workshop

Week 5 Workshop

Conceptual Review

(a) **Set Definitions**

Set Equality: $A = B := \forall x(x \in A \leftrightarrow x \in B)$
Subset: $A \subseteq B := \forall x(x \in A \rightarrow x \in B)$
Union: $A \cup B := \{x \ : \ x \in A \vee x \in B\}$
Intersection: $A \cap B := \{x \ : \ x \in A \wedge x \in B\}$
Set Difference: $A \setminus B = A - B := \{x \ : \ x \in A \wedge x \notin B\}$
Set Complement: $\overline{A} = A^C := \{x \ : \ x \notin A\}$
Powerset: $\mathcal{P}(A) := \{B \ : B \subseteq A\}$
Cartesian Product: $A \times B := \{(a, b) \ : a \in A, \ b \in B\}$

(b) How do we prove that for sets A and B, $A \subseteq B$?

Solution:

Let $x \in A$ be arbitrary... thus $x \in B$. Since x was arbitrary, $A \subseteq B$.

(c) How do we prove that for sets A and B, $A = B$?

Solution:

Use two subset proofs to show that $A \subseteq B$ and $B \subseteq A$.

1. Set Operations

Let $A = \{1, 2, 5, 6, 8\}$ and $B = \{2, 3, 5\}$.

(a) What is the set $A \cap (B \cup \{2, 8\})$?

Solution:

$\{2, 5, 8\}$

(b) What is the set $\{10\} \cup (A \setminus B)$?

Solution:

$\{1, 6, 8, 10\}$

(c) What is the set $\mathcal{P}(B)$?

Solution:

$\{\{2, 3, 5\}, \{2, 3\}, \{2, 5\}, \{3, 5\}, \{2\}, \{3\}, \{5\}, \emptyset\}$

(d) How many elements are in the set $A \times B$? List 3 of the elements.

Solution:

15 elements, for example $(1,2),(1,3),(1,5)$.

2. A Basic Subset Proof

Prove that $A \cap B \subseteq A \cup B$.

Solution:

Let $x \in A \cap B$ be arbitrary. By definition of intersection, $x \in A$ and $x \in B$. So, certainly $x \in A$ or $x \in B$. Then by definition of union, $x \in A \cup B$. Since x was arbitrary, $A \cap B \subseteq A \cup B$.

3. Set Equality Proof (Parts)

(a) Prove that $A \cap (A \cup B) \subseteq A$.

Solution:

Let x be an arbitrary element of $A \cap (A \cup B)$. By definition of intersection, $x \in A$ and $x \in A \cup B$. So certainly, $x \in A$. Since x was arbitrary, $A \cap (A \cup B) \subseteq A$.

(b) Prove that $A \subseteq A \cap (A \cup B)$.

Solution:

Let $x \in A$ be arbitrary. Since $x \in A$, certainly $x \in A$ or $x \in B$. By definition of union, $x \in A \cup B$. Since $x \in A$ and $x \in A \cup B$, by definition of intersection, $x \in A \cap (A \cup B)$. Since x was arbitrary, $A \subseteq A \cap (A \cup B)$.

(c) Combine part (a) and (b) to conclude that $A \cap (A \cup B) = A$.

Solution:

We have shown that $A \cap (A \cup B) \subseteq A$ and $A \subseteq A \cap (A \cup B)$. Therefore, we can conclude that $A \cap (A \cup B) = A$.

4. $\cup \to \cap$?

Prove or disprove: $A \cup B \subseteq A \cap B$.

Solution:

We disprove the claim by finding a counter-example. Consider $A = \{1,2\}$ and $B = \{1,3\}$.
$A \cup B = \{1,2,3\}$, and $A \cap B = \{1\}$. $\{1,3\} \not\subseteq \{1\}$, so the claim does not hold for these choices of A, B. Since we have found a counter-example, we have shown that it is not the case that $A \cup B \subseteq A \cap B$ for all sets A and B.

5. Subsets

Prove or disprove: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Solution:

Let A, B, C be arbitrary sets. Suppose $A \subseteq B$ and $B \subseteq C$. Let x be an arbitrary element of A. By definition of subset, $x \in B$, and by definition of subset again, $x \in C$. Since x was an arbitrary element of A, $A \subseteq C$.

6. Set Equality Proof

Prove that $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Solution:

First, we show $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$.
Let x be an arbitrary element of $A \setminus (B \cap C)$.
By definition of set difference, $x \in A \wedge \neg(x \in B \cap C)$.
By definition of intersection, $x \in A \wedge \neg(x \in B \wedge x \in C)$.
Applying DeMorgan's Law, we get $x \in A \wedge (x \notin B \vee x \notin C)$.
Applying the distributive law, we get $(x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C)$.
Then, applying the definition of set difference twice, get $x \in (A \setminus B) \vee x \in (A \setminus C)$.
By definition of union, we get $x \in (A \setminus B) \cup (A \setminus C)$.
Since x was arbitrary, $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$.

Now we show $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$.
Let x be an arbitrary element of $(A \setminus B) \cup (A \setminus C)$.
By definition of union, $x \in (A \setminus B) \vee x \in (A \setminus C)$.
Applying the definition of set difference twice, we get $(x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C)$.
Applying the distributive law, we get $(x \in A) \wedge (x \notin B \vee x \notin C)$.
By DeMorgan's Law, we have $(x \in A) \wedge \neg(x \in B \wedge x \in C)$.
By definition of intersection, we have $(x \in A) \wedge \neg(x \in B \cap C)$.
By definition of set difference, $x \in A \setminus (B \cap C)$.
Since x was arbitrary, $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$.

Combining these two directions, since both sets are subsets of each other, we have shown $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.