CSE 390Z: Mathematics for Computation Workshop

Week 3 Workshop Problems Solutions

Conceptual Review

(a) Inference Rules:

Introduce $\vee$:	$\frac{A}{\therefore A \vee B,\ B \vee A}$	Eliminate $\vee$:	$\frac{A \vee B\ ;\ \neg A}{\therefore B}$
Introduce $\wedge$:	$\frac{A\ ;\ B}{\therefore A \wedge B}$	Eliminate $\wedge$:	$\frac{A \wedge B}{\therefore A\ ,\ B}$
Direct Proof:	$\frac{A \Rightarrow B}{\therefore A \to B}$	Modus Ponens:	$\frac{A\ ;\ A \to B}{\therefore B}$
Intro $\exists$:	$\frac{P(c) \text{ for some } c}{\therefore \exists x P(x)}$	Eliminate $\exists$:	$\frac{\exists x P(x)}{\therefore P(c) \text{ for a fresh } c}$
Intro $\forall$:	$\frac{P(a);\ a \text{ is arbitrary}}{\therefore \forall x P(x)}$	Eliminate $\forall$:	$\frac{\forall x P(x)}{\therefore P(a) \text{ for any } a}$

(b) Given $A \wedge B$, prove $A \vee B$

Solution:

1. $A \wedge B$ (Given)
2. A (Elim $\wedge$: 1.)
3. $A \vee B$ (Intro $\vee$: 2.)

(c) What is the purpose of the direct proof rule? How do you use it? Why are we allowed to do this?

Solution:

We use the direct proof rule to prove an implication (e.g., $p \to q$).

First, we assume the premise of the implication (e.g., p) is true. We indent this part of our solution to provide a visual indication that we're working under an assumption. Now that we've assumed the premise holds, we can use it the same way we would use any other expression in our proof. We then apply our inference rules and arrive at our conclusion (e.g., q). We step back outside our cozy indented assumption box and conclude that the implication $p \to q$ holds.

Now, why can we do this? When we indent our proof, we don't know if our premise p is true or not. We're not asserting that it is true. We're asking the reader to pretend for a second that p is true. When we use this assumption to show that q is true, do we know that q is actually true? NO! That conclusion is based on the assumption that p is true, and we made that up!! What we do know is that *if* p is true *then* q must be true, so we can conclude that $p \to q$.

(d) Given $P \to R$, $R \to S$, prove $P \to S$.

Solution:

1. $P \to R$ (Given)
2. $R \to S$ (Given)
   3.1 P (Assumption)

3.2 R (Modus Ponens: 3.1, 1)

3.3 S (Modus Ponens: 3.2, 2)

3. $P \to S$ (Direct Proof Rule)

(e) How do we prove a "for all" statement?

Solution:

In an Inference Proof:

1. Let a be arbitrary

   ...

5. $P(a)$
6. $\forall x(P(x))$ (Intro $\forall$: 1, 5)

In a Direct (English) Proof:
Let a be arbitrary.
...
So, $P(a)$ is true.
Since a was arbitrary, $P(a)$ is true for all a.

1. Translations with Integers

Translate the following English sentences to predicate logic. The **domain of discourse is integers**, and you may use $=$, $\neq$, and $>$ as predicates. Assume the predicates Prime, Composite, and Even have been defined appropriately.

Note: Composite numbers are ones that have more than 2 factors.

(a) 2 is prime.

Solution:

$\text{Prime}(2)$

(b) Every integer greater than 1 is prime or composite, but not both.

Solution:

$\forall x\ ((x > 1) \rightarrow (\text{Prime}(x) \oplus \text{Composite}(x)))$

OR

$\forall x\ ((x > 1) \rightarrow [(\text{Prime}(x) \wedge \neg\text{Composite}(x)) \vee (\neg\text{Prime}(x) \wedge \text{Composite}(x))])$

(c) There is **exactly one** even prime.

Solution:

$\exists x(\text{Even}(x) \wedge \text{Prime}(x) \wedge \forall y[(\text{Even}(y) \wedge \text{Prime}(y)) \rightarrow (y = x)])$

OR

$\exists x(\text{Even}(x) \wedge \text{Prime}(x) \wedge \forall y[(\text{Even}(y) \wedge y \neq x) \rightarrow \neg Prime(y)])$

OR

$\exists x \forall y[(y = x) \leftrightarrow (\text{Prime}(y) \wedge \text{Even}(y))]$*

(d) 2 is the only even prime.

Solution:

$\forall y\ [(y = 2) \leftrightarrow (\text{Prime}(y) \wedge \text{Even}(y))]$ *Notice the similarity to * in part (c).*

(e) Some, but not all, composite integers are even.

Solution:

$\exists x(\text{Composite}(x) \wedge \text{Even}(x)) \wedge \neg\forall x(\text{Composite}(x) \rightarrow \text{Even}(x))$

OR

$\exists x(\text{Composite}(x) \wedge \text{Even}(x)) \wedge \exists x(\text{Composite}(x) \wedge \neg\text{Even}(x))$

2. Inference Proofs: Modus Ponens

(a) Prove that given $p \to q$, $\neg s \to \neg q$, and p, we can conclude s. **Hint:** You may need to use a contrapositive at some point.

Solution:

1. $p \to q$ (Given)
2. $\neg s \to \neg q$ (Given)
3. p (Given)
4. q (Modus Ponens; 1,3)
5. $q \to s$ (Contrapositive; 2)
6. s (Modus Ponens; 5,4)

(b) Prove that given $\neg\ s \to (q \vee p)$, $\neg p$, and $\neg s$, we can conclude q.

Solution:

1. $\neg\ s \to (q \vee p)$ (Given)
2. $\neg p$ (Given)
3. $\neg s$ (Given)
4. $q \vee p$ (Modus Ponens; 1,3)
5. $p \vee q$ (Commutativity; 4)
6. q (Elim $\vee$; 5,2)

3. Inference Proofs: Direct Proof Rule

(a) Prove that given $p \to q$, we can conclude $(p \wedge r) \to q$

Solution:

1. $p \to q$ (Given)
   - 2.1 $p \wedge r$ (Assumption)
   - 2.2 p (Elim $\wedge$; 2.1)
   - 2.3 q (Modus Ponens; 1, 2.2)
2. $(p \wedge r) \to q$ (Direct proof rule)

(b) Prove that given $p \vee q$, $q \to r$, and $r \to s$, we can conclude $\neg p \to s$.

Solution:

1. $p \vee q$ (Given)
2. $q \to r$ (Given)
3. $r \to s$ (Given)
   - 4.1 $\neg p$ (Assumption)
   - 4.2 q (Elim $\vee$; 1, 4.1)

4.3 r (Modus Ponens; 2, 4.2)

4.4 s (Modus Ponens; 3, 4.3)

4. $\neg p \rightarrow s$ (Direct proof rule)

4. Inference Proofs with Quantifiers

(a) Prove that $\forall x P(x) \rightarrow \exists x P(x)$. You may assume that the domain is nonempty.

Solution:

1.1. $\forall x P(x)$ (Assumption)

1.2. $P(a)$ (Elim $\forall$: 1.1)

1.3. $\exists x P(x)$ (Intro $\exists$: 1.2)

2. $\forall x P(x) \rightarrow \exists x P(x)$ (Direct Proof Rule)

(b) Given $\forall x(T(x) \rightarrow M(x))$ and $\exists x(T(x))$, prove that $\exists x(M(x))$.

Solution:

1. $\forall x(T(x) \rightarrow M(x))$ (Given)
2. $\exists x(T(x))$ (Given)
3. $T(r)$* (Elim $\exists$; 2)
4. $T(r) \rightarrow M(r)$** (Elim $\forall$; 1)
5. $M(r)$ (Modus Ponens; 4, 3)
6. $\exists x(M(x))$ (Intro $\exists$; 5)

* r is the value that satisfies $T(x)$

** The Eliminate $\forall$ rule says: Eliminate $\forall$: $\frac{\forall x P(x)}{\therefore P(a) \text{ for any } a}$. We can pick any value in the domain for a, so we pick the r from step 3.

(c) Given $\forall x(P(x) \rightarrow Q(x))$, prove that $(\exists x P(x)) \rightarrow (\exists y Q(y))$.

Solution:

1. $\forall x(P(x) \rightarrow Q(x))$ (Given)

2.1. $\exists x(P(x))$ (Assumption)

2.2. $P(r)$* (Elim $\exists$; 2.1)

2.3. $P(r) \rightarrow Q(r)$** (Elim $\forall$; 1)

2.4. $Q(r)$ (Modus Ponens; 2.3, 2.2)

2.5. $\exists y(Q(y))$ (Intro $\exists$; 2.4)

2. $(\exists x P(x)) \rightarrow (\exists y Q(y))$ (Direct Proof Rule)

* r is the value that satisfies $P(x)$

** The Eliminate $\forall$ rule says: Eliminate $\forall$: $\frac{\forall x P(x)}{\therefore P(a) \text{ for any } a}$. We can pick any value in the domain for a, so we pick the r from step 2.2.

5. Direct Proof

Let the predicates Odd(x) and Even(x) be defined as follows where the domain of discourse is integers:

$$\text{Odd}(x) := \exists k\ (x = 2k + 1)$$

$$\text{Even}(x) := \exists k\ (x = 2k)$$

Prove the following claim using a **Direct Proof** written as an English proof (i.e., using the Direct Proof template you saw in lecture).

$$\forall x \forall y[(Even(x) \wedge Odd(y)) \rightarrow Odd(x + y)]$$

(a) Declare an arbitrary variable for each $\forall$.

Solution:

Let x and y be arbitrary integers.

(b) Assume the left side of the implication.

Solution:

Suppose x is even and y is odd.

(c) Unroll the predicate definitions from your assumptions.

Solution:

Then by definition of even, there exists some integer k such that $x = 2k$, and by definition of odd, there exists some integer j such that $y = 2j + 1$.
Note: A common mistake here is to declare k and j as arbitrary. They're not arbitrary – they're the specific integers that satisfy the equations $x = 2k$ and $y = 2j + 1$.

(d) Manipulate what you have towards your goal.

Solution:

Adding x and y, we see that: $x + y = (2k) + (2j + 1) = 2k + 2j + 1 = 2(k + j) + 1$.

(e) Reroll definitions into the right side of the implication.

Solution:

Since k and j are both integers, $k + j$ is also an integer. So $x + y$ is two times and integer plus 1. By definition of odd, $x + y$ is odd.

(f) Conclude that you have proved the claim.

Solution:

Because x and y were arbitrary integers, we can conclude that for all integers x and y, if x is even and y is odd then $x + y$ is odd.

(g) Now take these proof parts and assemble them into one cohesive Direct proof.

Solution:

Let x and y be arbitrary integers. Suppose x is even and y is odd. Then by definition of even, there exists some integer k such that $x = 2k$, and by definition of odd, there exists some integer j such that $y = 2j + 1$. Adding x and y we see that: $x + y = (2k) + (2j + 1) = 2k + 2j + 1 = 2(k + j) + 1$. Since k and j are both integers, $k + j$ is also an integer. So $x + y$ is two times and integer plus 1. By definition of odd, $x + y$ is odd. Because x and y were both arbitrary integers, we can conclude that for all integers x and y, if x is even and y is odd, then $x + y$ is odd.

6. Oddly Even

(a) Let the predicates Odd(x) and Even(x) be defined as follows where the domain of discourse is integers:

$$\text{Odd}(x) := \exists k\ (x = 2k + 1)$$
$$\text{Even}(x) := \exists k\ (x = 2k)$$

Write an **Inference Proof** of the following claim:

$$\forall x \forall y[(Odd(x) \wedge Odd(y)) \rightarrow Even(x + y)]$$

Hint: You can cite the definitions of Even and Odd as well as "Algebra" to justify your steps the same way you cite Inference Rules to justify your steps.
Hint: There aren't any "Givens" here. Start by introducing two arbitrary integers a and b, then show that $(Odd(a) \wedge Odd(b)) \rightarrow Even(a + b)$, then use the "Intro $\forall$" rule twice.

Solution:

1. Let a be an arbitrary integer.
2. Let b be an arbitrary integer.
   - 3.1. Odd$(a) \wedge$ Odd(b) [Assumption]
   - 3.2. Odd(a) [Elim $\wedge$: 3.1]
   - 3.3. $\exists k\ (a = 2k + 1)$ [Definition of Odd, 3.2]
   - 3.4. $a = 2m + 1$ [Elim $\exists$: 3.3]
   - 3.5. Odd(b) [Elim $\wedge$: 3.1]
   - 3.6. $\exists k\ (b = 2k + 1)$ [Definition of Odd, 3.5]
   - 3.7. $b = 2j + 1$ [Elim $\exists$: 3.7]
   - 3.8. $a + b = 2m + 1 + 2j + 1$ [Algebra: 3.4, 3.7]
   - 3.9. $a + b = 2(m + j + 1)$ [Algebra: 3.8]
   - 3.10. $\exists k\ (a + b = 2k)$ [Intro $\exists$: 3.9]
   - 3.11. Even$(a + b)$ [Definition of Even, 3.10]
3. Odd$(a) \wedge$ Odd$(b) \rightarrow$ Even$(a + b)$ [Direct Proof Rule]
4. $\forall y$(Odd$(a) \wedge$ Odd$(y) \rightarrow$ Even$(a + y)$) [Intro $\forall$: 2,3]
5. $\forall x \forall y$(Odd$(x) \wedge$ Odd$(y) \rightarrow$ Even$(x + y)$) [Intro $\forall$: 1,4]

7. Challenge: Inference Proof

Given $\forall x\ (P(x) \vee Q(x))$ and $\forall y\ (\neg Q(y) \vee R(y))$, prove $\exists x\ (P(x) \vee R(x))$. You may assume that the domain is not empty.

Hint: You can cite logical equivalence rules too.

Solution:

1.	$\forall x\ (P(x) \vee Q(x))$	[Given]
2.	$\forall y\ (\neg Q(y) \vee R(y))$	[Given]
3.	$P(a) \vee Q(a)*$	[Elim $\forall$: 1]
4.	$\neg Q(a) \vee R(a) **$	[Elim $\forall$: 2]
5.	$Q(a) \to R(a)$	[Law of Implication: 4]
6.	$\neg\neg P(a) \vee Q(a)$	[Double Negation: 3]
7.	$\neg P(a) \to Q(a)$	[Law of Implication: 6]
	8.1. $\neg P(a)$ [Assumption]	
	8.2. $Q(a)$ [Modus Ponens: 7, 8.1]	
	8.3. $R(a)$ [Modus Ponens: 5, 8.2]	
8.	$\neg P(a) \to R(a)$	[Direct Proof Rule]
9.	$\neg\neg P(a) \vee R(a)$	[Law of Implication: 8]
10.	$P(a) \vee R(a)$	[Double Negation: 9]
11.	$\exists x\ (P(x) \vee R(x))$	[Intro $\exists$: 10]

* We can pick any value we want. We don't need anything special, so we pick a random thing in the domain and call it a.

** We can pick any value we want. We intentionally choose the same value we chose in step 3.