CSE 390Z: Mathematics for Computation Workshop

QuickCheck: Structural Induction Solutions

Please submit a response to the following questions on Gradescope. We do not grade on accuracy, so please submit your best attempt. You may either typeset your responses or hand-write them. Note that hand-written solutions must be legible to be graded.

We have created **this template** if you choose to typeset with Latex. **This guide** has specific information about scanning and uploading pdf files to Gradescope.

0. How Many Ones?

The set T is defined as follows:

- Base case: $\epsilon \in T$
- Recursive Rules:
  If $x \in T$, then $11x \in T$
  If $x \in T$ and $y \in T$, then $x0y \in T$

Given the following recursively defined function

- numOnes(ϵ) = 0
- numOnes($11x$) = 2 + numOnes(x)
- numOnes($x0y$) = numOnes(x) + numOnes(y)

Prove that for all strings n in T, numOnes(n) is even

Hint: In structural induction, the structure of your induction mirrors the recursive definition.

Solution:

Let $P(n)$ be "2 | numOnes(n)". We will show that $P(n)$ is true for all $n \in T$ by structural induction.

Base Case ($n = \epsilon$):
numOnes(ϵ) = 0 definition of numOnes
$0 = 2 \cdot 0$ and $2|0$ by definition of divides.
Therefore P(0) holds true.

Induction Hypothesis: Suppose $P(x)$ and $P(y)$ are true for some arbitrary elements $x, y \in T$.

Induction Step:

Goal: Prove $P(11x)$ and $P(x0y)$

numOnes($11x$) = 2 + numOnes(x) by definition of numOnes. By the inductive hypothesis, 2 | numOnes(x). Therefore, by definition of divides numOnes(x) = $2z$ for some integer z. Thus,

$$\text{numOnes}(11x) = 2 + \text{numOnes}(x) = 2z + 2 = 2(z + 1)$$

Therefore, by definition of divides, 2 | numOnes($11x$). Therefore, P($11x$) holds.

numOnes($x0y$) = numOnes(x) + numOnes(y) by definition of numOnes. By the induction hypothesis, 2 | numOnes(x) and 2 | numOnes(y). Therefore, by definition of divides, numOnes(x) = 2z for some integer z and numOnes(y) = 2q for some integer q. Thus,

$$\text{numOnes}(x0y) = \text{numOnes}(x) + \text{numOnes}(y) = 2z + 2q = 2(z + q)$$

Therefore, by definition of divides, 2 | numOnes($x0y$). Therefore, P($x0y$) holds.

The result follows for all $n \in T$ by structural induction.

1. Video Solution

Watch **this video** on the solution **after** making an initial attempt. Then, answer the following questions.

(a) What is one thing you took away from the video solution?