CSE 390Z: Mathematics for Computation Workshop

Practice 311 Midterm Solutions

Name: ____________________________________

UW ID: ____________________________________

Instructions:

- This is a **simulated practice midterm**. You will **not** be graded on your performance on this exam.
- Nevertheless, please treat this as if it is a real exam. That means that you may not discuss with your neighbors, reference outside material, or use your devices during the next 60 minute period.
- If you get stuck on a problem, consider moving on and coming back later. In the actual exam, there will likely be opportunity for partial credit.
- There are 5 problems on this exam, totaling 80 points.

1. Predicate Translation [15 points]
Let the domain of discourse be novels, comic books, movies, and TV shows. Translate the following statements to predicate logic, using the following predicates:

Novel(x) := x is a novel
Comic(x) := x is a comic book
Movie(x) := x is a movie
Show(x) := x is a TV show
Adaptation(x, y) := x is an adaptation of y

(a) (5 points) A novel cannot be adapted into both a movie and a TV show.

Solution:

$\forall x(\text{Novel}(x) \rightarrow \forall m \forall s((\text{Movie}(m) \wedge \text{Show}(s)) \rightarrow \neg(\text{Adaptation}(m, x) \wedge \text{Adaptation}(s, x)))$

(b) (5 points) Every movie is an adaptation of a novel or a comic book.

Solution:

$\forall m(\text{Movie}(m) \rightarrow \exists x(\text{Adaptation}(m, x) \wedge (\text{Novel}(x) \vee \text{Comic}(x))))$

(c) (5 points) Every novel has been adapted into exactly one movie.

Solution:

$\forall x(\text{Novel}(x) \rightarrow \exists m(\text{Movie}(m) \wedge \text{Adaptation}(m, x) \wedge \forall n((\text{Movie}(n) \wedge (n \neq m)) \rightarrow \neg\text{Adaptation}(n, x))))$

OR

$\forall x(\text{Novel}(x) \rightarrow \exists m(\text{Movie}(m) \wedge \text{Adaptation}(m, x) \wedge \forall n(\text{Adaptation}(n, x) \rightarrow (\neg\text{Movie}(n) \vee n = m))))$

OR

$\forall x(\text{Novel}(x) \rightarrow \exists m(\text{Movie}(m) \wedge \text{Adaptation}(m, x) \wedge \forall n((\text{Adaptation}(n, x) \wedge \text{Movie}(n)) \rightarrow (n = m))))$

*Note that a great exercise is to show that the above 3 solutions are all logically equivalent :)

2. Boolean Algebra [15 points]

Let f be the boolean function defined as $f(x, y, z) = (x + y)' + (zy)$

(a) (5 points) Fill in the following table with the values of $f(x, y, z)$ in the last column. Feel free to use the blank columns while doing your work.

x	y	z				$f(x, y, z)$
0	0	0				
0	0	1				
0	1	0				
0	1	1				
1	0	0				
1	0	1				
1	1	0				
1	1	1				

Solution:

x	y	z	$(x+y)$	$(x+y)'$	(zy)	$f(x,y,z)$
0	0	0	0	1	0	1
0	0	1	0	1	0	1
0	1	0	1	0	0	0
0	1	1	1	0	1	1
1	0	0	1	0	0	0
1	0	1	1	0	0	0
1	1	0	1	0	0	0
1	1	1	1	0	1	1

(b) (5 points) Write $f(x, y, z)$ as a Boolean algebra expression in the canonical sum-of-products form in terms of the three input bits x, y, and z.

Solution:

$f(x, y, z) = x'y'z' + x'y'z + x'yz + xyz$

(c) (5 points) Write $f(x, y, z)$ as a Boolean algebra expression in the canonical product-of-sums form in terms of the three input bits x, y, and z.

Solution:

$f(x, y, z) = (x + y' + z)(x' + y + z)(x' + y + z')(x' + y' + z)$

3. **Formal Proof with Divides** [10 points]

Prove the following statement using a formal proof:

For all integers a, b, c, if $a^2 \mid b$ and $b^3 \mid c$, then $a^6 \mid c$.

Solution:

1. Let x, y, z be arbitrary integers.

2.1	$x^2 \mid y \wedge y^3 \mid z$	Assumption
2.2	$x^2 \mid y$	Elim $\wedge$: 2.1
2.3	$y^3 \mid z$	Elim $\wedge$: 2.2
2.4	$\exists k(y = x^2 k)$	Definition of divides: 2.2
2.5	$\exists k(z = y^3 k)$	Definition of divides: 2.3
2.6	$y = x^2 s$	Elim $\exists$: 2.4
2.7	$z = y^3 t$	Elim $\exists$: 2.5
2.8	$z = (x^2 s)^3 t = x^6(s^3 t)$	Algebra
2.9	$\exists k(z = x^6 k)$	Intro $\exists$: 2.8
2.10	$x^6 \mid z$	Definition of divides: 2.9

2. $x^2 \mid y \wedge y^3 \mid z \rightarrow x^6 \mid z$ — Direct Proof: 2.1 - 2.10
3. $\forall a \forall b \forall c((a^2 \mid b \wedge b^3 \mid c) \rightarrow a^6 \mid c$ — Intro $\forall$

4. English Proof with Mod [20 points]

Write an English proof to show that for all integers n, $n^2 \equiv_4 0$ or $n^2 \equiv_4 1$.

Hint: You'll need to break this proof into cases based on the following definitions:
Even(x):= ∃k(x = 2k)
Odd(x):= ∃k(x = 2k + 1)

Solution:
Let n be an arbitrary integer.

Case 1: n is even. Then $n = 2k$ for some integer k. Then $n^2 = (2k)^2 = 4k^2$. Since k is an integer, k^2 is an integer. So n^2 is 4 times an integer. Then by definition of divides, $4 \mid n^2 - 0$. Then by definition of congruence, $n^2 \equiv_4 0$.

Case 2: n is odd. Then $n = 2k + 1$ for some integer k. Then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$. So $n^2 - 1 = 4(k^2 + k)$. Since k is an integer, $k^2 + k$ is an integer. So $n^2 - 1$ is 4 times an integer. Then by definition of divides, $4 \mid n^2 - 1$. Then by definition of congruence, $n^2 \equiv_4 1$.

Thus in all cases, $n^2 \equiv_4 0$ or $n^2 \equiv_4 1$. Since n was arbitrary, the claim holds.

5. Induction [20 points]
Prove by induction that $(1+\pi)^n > 1 + n\pi$ for all integers $n \geq 2$.

Solution:

1. Let P(n) be the statement "$(1+\pi)^n > 1 + n\pi$". We prove P(n) for all integers $n \geq 2$ by induction.

2. Base Case: When $n = 2$, the LHS is $(1+\pi)^2 = 1 + 2\pi + \pi^2$. The RHS is $1 + 2\pi$. Since $\pi^2 > 0$, $1 + 2\pi + \pi^2 > 1 + 2\pi$, so the Base Case holds.

3. Inductive Hypothesis: Suppose that P(k) holds for some arbitrary integer $k \geq 2$. Then $(1+\pi)^k > 1 + k\pi$.

4. Inductive Step:

$$\boxed{\text{Goal: Show } P(k+1)\text{, i.e. show } (1+\pi)^{k+1} > 1 + (k+1)\pi}$$

$$\begin{aligned}
(1+\pi)^{k+1} &= (1+\pi)(1+\pi)^k && \text{Definition of Exponent} \\
&> (1+\pi)(1+k\pi) && \text{By IH} \\
&= 1 + \pi + k\pi + k\pi^2 && \text{Algebra} \\
&= 1 + (k+1)\pi + k\pi^2 && \text{Algebra} \\
&> 1 + (k+1)\pi && \text{Since } k\pi^2 > 0
\end{aligned}$$

Thus $(1+\pi)^{k+1} > 1 + (k+1)\pi$. So P($k+1$) holds.

5. Thus we have proven P(n) for all integers $n \geq 2$ by induction.