CSE 390Z: Mathematics for Computation Workshop

Practice 311 Final Solutions

Name: ______________________________

UW ID: ______________________________

Instructions:

- This is a **simulated practice final**. You will **not** be graded on your performance on this exam.
- Nevertheless, please treat this as if it is a real exam. That means that you may not discuss with your neighbors, reference outside material, or use your devices during the next 60 minute period.
- If you get stuck on a problem, consider moving on and coming back later. In the actual exam, there will likely be opportunity for partial credit.
- There are 5 problems on this exam, totaling 100 points.

1. Language Representation (20 points)

Let the language L consist of all binary strings that do not contain 111.

(a) [5 points] Write a regular expression that represents L.

Solution:

$(0 \cup 10 \cup 110)^*(1 \cup 11 \cup \epsilon)$

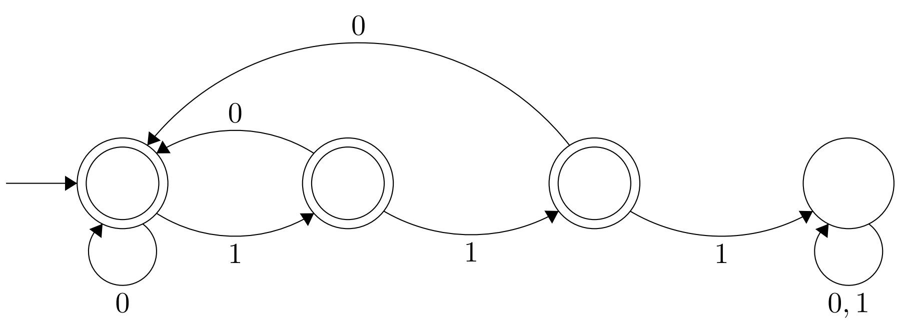
(b) [5 points] Write a CFG that generates all strings in L.

Solution:

S $\rightarrow$ **XY**
X $\rightarrow$ 0**X** | 10**X** | 110**X** | ϵ
Y $\rightarrow$ 1 | 11 | ϵ

(c) [5 points] Draw a DFA that accepts exactly the strings in L.

Solution:



(d) [5 points] Convert the following regular expression to a CFG:

$$10(0^* \cup 1^*)01$$

Solution:

S $\rightarrow$ 10**X**01 | 10**Y**01
X $\rightarrow$ 0**X** | ϵ
Y $\rightarrow$ 1**Y** | ϵ

2. Sets (20 points)

(a) [12 points] Prove using the Meta Theorem that for any sets A and B,

$$A \setminus \overline{B} = (A \cup \varnothing) \cap B$$

Hint 1: The empty set, $\varnothing$, is the set that contains no elements. i.e. $\varnothing ::= \{x : F\}$.
Hint 2: If you get stuck, try working backwards!

Solution:

The claim is equivalent to $\forall x(x \in (A \setminus \overline{B}) \leftrightarrow x \in ((A \cup \varnothing) \cap B))$.
Let x be arbitrary.

$$
\begin{aligned}
x \in (A \setminus \overline{B}) &\equiv x \in A \wedge \neg(x \in \overline{B}) && \text{Def of set difference} \\
&\equiv x \in A \wedge \neg\neg(x \in B) && \text{Def of complement} \\
&\equiv x \in A \wedge x \in B && \text{Double Negation} \\
&\equiv (x \in A \vee F) \wedge x \in B && \text{Identity} \\
&\equiv (x \in A \vee x \in \varnothing) \wedge x \in B && \text{Def of } \varnothing \\
&\equiv x \in (A \cup \varnothing) \wedge x \in B && \text{Def of union} \\
&\equiv x \in (A \cup \varnothing) \cap B && \text{Def of intersection}
\end{aligned}
$$

Since x was arbitrary, we have shown that these sets contain the same elements and are therefore equal.

Determine if the following claims are true or false. Then explain your reasoning in 1-3 sentences.
You may include images or examples in your explanation. **You do not need to give a formal proof or disproof.**

(b) [4 points] For all sets A, B: $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$.

Solution:

True. Both sets represent the set of elements that are in A or B but not in both. That is, both sets are equal to the set $\{x : x \in A \oplus x \in B\}$.

(c) [4 points] For all sets A, B: $\mathcal{P}(A) \times \mathcal{P}(B) \subseteq \mathcal{P}(A \times B)$.

Solution:

False. Consider $A = \{1\}$ and $B = \{2\}$. Then $(\{1\}, \{2\}) \in \mathcal{P}(A) \times \mathcal{P}(B)$ but $(\{1\}, \{2\}) \notin \mathcal{P}(A \times B)$.

3. Induction I (20 points)

Let the function $f : \mathbb{N} \to \mathbb{N}$ be defined as follows:

$f(0) = 2$
$f(1) = 7$
$f(n) = f(n-1) + 2f(n-2)$ for $n \geq 2$
Prove that $f(n) = 3 * 2^n + (-1)^{n+1}$ for all integers $n \geq 0$ using strong induction.

Solution:
Let $P(n)$ be "$f(n) = 3 * 2^n + (-1)^{n+1}$. We will prove $P(n)$ holds for all $n \geq 0$ by strong induction.

Base Cases:
$n = 0$:
$f(0) = 2$
$3 * 2^0 + (-1)^{0+1} = 3 * 1 + (-1)^1 = 3 + (-1) = 2.$
$2 = 2$, so $P(0)$ holds.

$n = 1$:
$f(1) = 7$
$3 * 2^1 + (-1)^{1+1} = 3 * 2 + (-1)^2 = 6 + 1 = 7$
$7 = 7$, so $P(1)$ holds.

Inductive Hypothesis: Suppose that $P(j)$ holds for all $0 \leq j \leq k$ for some arbitrary integer k.
Inductive Step:

Goal: Show $P(k+1)$, i.e. show $f(k+1) = 3 * 2^{k+1} + (-1)^{(k+1)+1}$

$$\begin{aligned}
f(k+1) &= f((k+1)-1) + 2f((k+1)-2) && \text{def. of } f \\
&= f(k) + 2f(k-1) \\
&= 3 * 2^k + (-1)^{k+1} + 2(3 * 2^{k-1} + (-1)^{(k-1)+1}) && \text{IH} \\
&= 3 * 2^k + (-1)^{k+1} + 2(3 * 2^{k-1} + (-1)^k) \\
&= 3 * 2^k + (-1)^{k+1} + 3 * 2^k + 2(-1)^k \\
&= 2 * 3 * 2^k + (-1)^{k+1} + 2(-1)^k \\
&= 3 * 2^{k+1} + (-1)^{k+1} + 2(-1)^k \\
&= 3 * 2^{k+1} + (-1)(-1)(-1)^{k+1} + 2(-1)(-1)(-1)^k && (-1)(-1) = 1 \\
&= 3 * 2^{k+1} - (-1)^{k+2} + 2(-1)^{k+2} \\
&= 3 * 2^{k+1} + (-1)^{(k+1)+1}
\end{aligned}$$

Thus, $P(k+1)$ holds.

We conclude that $P(n)$ holds for all $n \geq 0$ by strong induction.

4. Induction II (20 points)

Let the set S be recursively defined as follows:
Basis: $(0, 0) \in S$
Recursive Step: If $(x, y) \in S$, then $(x + 2, y + 4) \in S$ and $(x + 4, y + 8) \in S$.

Prove that for all $(x, y) \in S$, $x + y$ is divisible by 3.

Solution:
Define $P((x, y))$ to be the claim $3 \mid (x + y)$. We will prove that $P((x, y))$ holds for all $(x, y) \in S$ by structural induction.
Base Case: $(x, y) = (0, 0)$
$0 + 0 = 0 = 3 * 0$. So, $3|(x + y)$ and $P((0, 0))$ holds.

Inductive Hypothesis: Suppose that $P((a, b))$ holds for some arbitrary $(a, b) \in S$. (i.e. $3|(a + b)$).

Inductive Step:

$$\boxed{\text{Goal: Show } P((a+2, b+4)) \text{ and } P((a+4, b+8))}$$

By the inductive hypothesis, $3 \mid (a + b)$. By definition of divides, $3k = a + b$ for some integer k.

$$(a + 2) + (b + 4) = a + b + 6 = 3k + 6 = 3(k + 2)$$

So, $3 \mid ((a + 2) + (b + 4))$ which means $P((a + 2, b + 4))$ holds.

$$(a + 4) + (b + 8) = a + b + 12 = 3k + 12 = 3(k + 4)$$

So, $3 \mid ((a + 4) + (b + 8))$ which means $P((a + 4, b + 8))$ holds.

Thus, $P((x, y))$ holds for all $(x, y) \in S$ by the principle of structural induction.

5. Irregularity (20 points)
Prove that the language $L = \{10^x10^{x+1}1 : x \geq 0\}$ is not regular.

Solution:
Suppose for the sake of contradiction there exists a DFA D that accepts L.

Let $S = \{10^n1 : n \geq 0\}$. Since S contains infinitely number strings and D has a finite number of states, two strings in S must end up in the same state of D. Say those strings are 10^i1 and 10^j1 for some $i, j \geq 0$ where $i \neq j$. Now, append $0^{i+1}1$ to both strings. The resulting strings are:

$x = 10^i10^{i+1}1$. Note that $x \in L$.
$y = 10^j10^{i+1}1$. Note that $y \notin L$ since $i \neq j, j + 1 \neq i + 1$.

Both x and y must end up at the same state, but since $x \in L$ and $y \notin L$, that state must be both an accept state and a reject state. This is a contradiction, which means there does not exist a DFA D which accepts L. This means L is not regular.