CSE 390Z: Mathematics for Computation Workshop

Mid-Quarter Review Solutions

Name: ______________________________

0. Training Wheels

For this problem, our domain of discourse is college football teams and college football conferences.
You are allowed to use the $\neq$ symbol to check that two objects are not equivalent.
We will use the following predicates:

- $\texttt{Team}(x) \coloneqq x$ is a football team.
- $\texttt{UW}(x) \coloneqq x$ is the University of Washington football team.
- $\texttt{WSU}(x) \coloneqq x$ is the Washington State University football team.
- $\texttt{OSU}(x) \coloneqq x$ is the Oregon State University football team.
- $\texttt{OldPac}(x) \coloneqq x$ is the old Pac-12 Conference.
- $\texttt{NewPac}(x) \coloneqq x$ is the new Pac-2 Conference.
- $\texttt{Member}(x, y) \coloneqq$ the football team x has been a part of the conference y.
- $\texttt{Lost}(x, y) \coloneqq x$ lost to y in a football game.

(a) State whether the two statements below are equivalent. Provide a one sentence justification.

$$\exists y \Big[\texttt{OldPac}(y) \land \forall x \Big(\texttt{Team}(x) \to \big(\texttt{UW}(x) \to \texttt{Member}(x, y)\big)\Big)\Big]$$
$$\exists y \Big[\texttt{OldPac}(y) \land \forall x \Big(\texttt{UW}(x) \to \texttt{Member}(x, y)\Big)\Big]$$

Solution:

Yes, they are equivalent. The UW Huskies are already a football team, so there is no need to actually state $\texttt{Team}(x)$.

(b) Translate the following sentence into predicate logic.

Excluding WSU, at least one team has been a part of the new Pac-2 conference and the old Pac-12 conference.

Solution:

$$\exists x \exists y \exists z \Big[\texttt{Team}(x) \land \neg\texttt{WSU}(x) \land \texttt{OldPac}(y) \land \texttt{Member}(x, y) \land \texttt{NewPac}(z) \land \texttt{Member}(x, z)\Big]$$

(c) Translate the following statement into predicate logic.

UW has won against all football teams besides itself, and WSU has lost to all foootball teams besides itself.

Solution:

$$\exists x \exists y \Big[\texttt{UW}(x) \land \forall a \big((\texttt{Team}(a) \land (a \neq x)) \rightarrow \texttt{Lost}(a,x) \big) \land$$
$$\texttt{WSU}(y) \land \forall b \big((\texttt{Team}(b) \land (b \neq y)) \rightarrow \texttt{Lost}(y,b) \big) \Big]$$

(d) Negate the following statement. Your final answer should have zero negations.

Warning: this statement makes absolutely no sense. Do **NOT** spend time thinking about its meaning. We want you to blindly follow your equivalency laws here.

$$\forall x \forall y \Big[\big(\texttt{WSU}(x) \land \texttt{OSU}(y)\big) \land \big(\neg\texttt{Lost}(x,y) \lor \neg\texttt{Lost}(y,x)\big) \Big]$$

Solution:

Students only need to write the final answer to receive full credit.

$$\exists x \exists y \Big[\big(\texttt{WSU}(x) \land \texttt{OSU}(y)\big) \rightarrow \big(\texttt{Lost}(x,y) \land \texttt{Lost}(y,x)\big) \Big]$$

The corresponding chain of equivalences is provided below for reference.

$$\neg\forall x \forall y \Big[\big(\texttt{WSU}(x) \land \texttt{OSU}(y)\big) \land \big(\neg\texttt{Lost}(x,y) \lor \neg\texttt{Lost}(y,x)\big) \Big]$$

$$\equiv \neg\forall x \forall y \Big[\big(\texttt{WSU}(x) \land \texttt{OSU}(y)\big) \land \neg\big(\texttt{Lost}(x,y) \land \texttt{Lost}(y,x)\big) \Big] \quad \text{DeMorgans}$$

$$\equiv \exists x \exists y \neg \Big[\big(\texttt{WSU}(x) \land \texttt{OSU}(y)\big) \land \neg\big(\texttt{Lost}(x,y) \land \texttt{Lost}(y,x)\big) \Big] \quad \text{DeMorgans for Quantifiers}$$

$$\equiv \exists x \exists y \Big[\neg\big(\texttt{WSU}(x) \land \texttt{OSU}(y)\big) \lor \neg\neg\big(\texttt{Lost}(x,y) \land \texttt{Lost}(y,x)\big) \Big] \quad \text{DeMorgans}$$

$$\equiv \exists x \exists y \Big[\neg\big(\texttt{WSU}(x) \land \texttt{OSU}(y)\big) \lor \big(\texttt{Lost}(x,y) \land \texttt{Lost}(y,x)\big) \Big] \quad \text{Double Negation}$$

$$\equiv \exists x \exists y \Big[\big(\texttt{WSU}(x) \land \texttt{OSU}(y)\big) \rightarrow \big(\texttt{Lost}(x,y) \land \texttt{Lost}(y,x)\big) \Big] \quad \text{Law of Implication}$$

1. Normal Forms

Consider the following function F:

p	q	r	$F(p,q,r)$
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	T

(a) Write a propositional logic expression for F in DNF form (ORs of ANDs).

Solution:

$$(p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$$

(b) Write a propositional logic expression for F in CNF form (ANDs of ORs).

Solution:

$$(\neg p \vee \neg q \vee \neg r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee q \vee \neg r)$$

2. Modular Arithmetic

Prove that for all integers $x, y, n > 0$, if $x \equiv_{6n} 1$ and $y \equiv_{7n} 5$ then $7x + 2y \equiv_{14n} 17$.

Hint: Apply the definition of congruence and divides.

Solution:

Let $x, y, n > 0$ be arbitrary integers. Suppose $x \equiv_{6n} 1$ and $y \equiv_{7n} 5$. Then by definition of congruence, $6n \mid (x-1)$ and $7n \mid (y-5)$. Then by definition of divides, there exists integers j, k such that $x - 1 = 6nk$ and $y - 5 = 7nj$. Thus $x = 6nk + 1$ and $y = 7nj + 5$. Then observe:

$$\begin{aligned} 7x + 2y &= 7(6nk + 1) + 2(7nj + 5) \\ &= 42nk + 7 + 14nj + 10 \\ &= 42nk + 14nj + 17 \\ &= 14n(3k + j) + 17 \end{aligned}$$

Then $(7x + 2y) - 17 = 14n(3k + j)$. Since k, j are integers, $3k + j$ is an integer. So by definition of divides, $14n \mid (7x + 2y) - 17$. Then by definition of congruence, $7x + 2y \equiv_{14n} 17$. Since x, y, n were arbitrary, the claim holds.

3. Extended Euclidean Algorithm

Find all solutions in the range of $0 \leq x < 2021$ to the modular equation:

$$311x \equiv_{2021} 3$$

Solution:

$$\begin{aligned}\text{gcd}(2021, 311) &= \text{gcd}(311, 2021 \text{ mod } 311) = \text{gcd}(311, 155) \\ &= \text{gcd}(155, 311 \text{ mod } 155) = \text{gcd}(155, 1) \\ &= 1\end{aligned}$$

Then we know that there is a multiplicative inverse:

$$\begin{aligned}2021 &= 311 * 6 + 155 \\ 311 &= 155 * 2 + 1 \\ 155 &= 1 * 155\end{aligned}$$

From here, we can rearrange the equations to get:

$$\begin{aligned}155 &= 2021 - 311 * 6 \\ 1 &= 311 - 155 * 2\end{aligned}$$

From here, we use back substitution and plug these back into our equations:

$$\begin{aligned}1 &= 311 - 155 * 2 \\ 1 &= 311 - 2 * (2021 - 311 * 6) \\ 1 &= 311 - 2 * 2021 + 12 * 311 \\ 1 &= 13 * 311 - 2 * 2021\end{aligned}$$

So the multiplicative inverse is 13, i.e. $311 * 13 \equiv_{2021} 1$. We can then multiply both sides of the original modular equation by 13 to get $13 * 311x \equiv_{2021} 13 * 3$. Simplifying gives us $x \equiv_{2021} 39$. By the definition of congruence and division we have $x = 39 + 2021k$ for $k \in \mathbb{N}$, but since we're only asked for solutions in the range of $0 \leq x < 2021$, $x = 39$.

4. Induction

Prove by induction that $3^n - 1$ is divisible by 2 for any integer $n \geq 1$.

Solution:

1. Let P(n) be the statement "$3^n - 1$ is divisible by 2". We prove P(n) for all integers $n \geq 1$ by induction.

2. Base Case: When $n = 1$, $3^n - 1 = 3^1 - 1 = 3 - 1 = 2$. Since $2 \mid 2$, the base case holds.

3. Inductive Hypothesis: Suppose that P(k) holds for some arbitrary integer $k \geq 1$. Then $2 \mid 3^k - 1$. Then by definition of divides, there exists some integer a such that $3^k - 1 = 2a$.

4. Inductive Step: Observe that...

$$
\begin{aligned}
3^{k+1} - 1 &= 3(3^k) - 1 && \text{Definition of Exponent} \\
&= 3(3^k - 1 + 1) - 1 && \text{Subtract and Add by 1} \\
&= 3(2a + 1) - 1 && \text{By IH} \\
&= 6a + 3 - 1 && \text{Algebra} \\
&= 6a + 2 && \text{Algebra} \\
&= 2(3a + 1) && \text{Algebra}
\end{aligned}
$$

Thus by definition of divides, $2 \mid 3^{k+1} - 1$. So P$(k+1)$ holds.

5. Thus we have proven P(n) for all integers $n \geq 1$ by induction.

5. Strong Induction

Consider the function f, which takes a natural number as input and outputs a natural number.

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ f(n-1) + 2 \cdot f(n-2) & \text{if } n \geq 2 \end{cases}$$

Prove that $f(n) = 2^n$ for all $n \in \mathbb{N}$.

Solution:

Let $P(n)$ be the claim that $f(n) = 2^n$. We will prove $P(n)$ true for all $n \in \mathbb{N}$ by strong induction.

- $f(0) = 1 = 1 = 2^0$ so $P(0)$ holds.
- $f(1) = 2 = 2 = 2^1$ so $P(1)$ holds.

Suppose that for some arbitrary integer $k \geq 1$, that $P(j)$ holds for all $j \in \mathbb{N}$ such that $j \leq k$.
Show $P(k+1)$, i.e. $f(k+1) = 2^{k+1}$

$$\begin{aligned} f(k+1) &= f(k+1-1) + 2 \cdot f(k+1-2) && \text{Definition of } f \\ &= f(k) + 2 \cdot f(k-1) \\ &= 2^k + 2 \cdot 2^{k-1} && \text{I.H. twice} \\ &= 2^k + 2^{k-1+1} \\ &= 2^k + 2^k \\ &= 2 \cdot 2^k \\ &= 2^{k+1} \end{aligned}$$

Clearly, $P(k+1)$ holds.
Therefore, we have proven the claim true by strong induction.