CSE 390Z: Mathematics for Computation Workshop

Week 8 Workshop Solutions

Conceptual Review

(a) **Set Definitions**

Set Equality: $A = B := \forall x(x \in A \leftrightarrow x \in B)$
Subset: $A \subseteq B := \forall x(x \in A \rightarrow x \in B)$
Union: $A \cup B := \{x \ : \ x \in A \vee x \in B\}$
Intersection: $A \cap B := \{x \ : \ x \in A \wedge x \in B\}$
Set Difference: $A \setminus B = A - B := \{x \ : \ x \in A \wedge x \notin B\}$
Set Complement: $\overline{A} = A^C := \{x \ : \ x \notin A\}$
Powerset: $\mathcal{P}(A) := \{B \ : B \subseteq A\}$
Cartesian Product: $A \times B := \{(a, b) \ : a \in A, \ b \in B\}$

(b) How do we prove that for sets A and B, $A \subseteq B$?

Solution:

Let $x \in A$ be arbitrary... thus $x \in B$. Since x was arbitrary, $A \subseteq B$.

(c) How do we prove that for sets A and B, $A = B$?

Solution:

Use two subset proofs to show that $A \subseteq B$ and $B \subseteq A$.

1. A Basic Subset Proof

Prove that $A \cap B \subseteq A \cup B$.

Solution:

Let $x \in A \cap B$ be arbitrary. Then by definition of intersection, $x \in A$ and $x \in B$. So certainly $x \in A$ or $x \in B$. Then by definition of union, $x \in A \cup B$.

2. Set Equality Proof

(a) Write an English proof to show that $A \cap (A \cup B) \subseteq A$ for any sets A, B.

Solution:

Let x be an arbitrary member of $A \cap (A \cup B)$. Then by definition of intersection, $x \in A$ and $x \in A \cup B$. So certainly, $x \in A$. Since x was arbitrary, $A \cap (A \cup B) \subseteq A$.

(b) Write an English proof to show that $A \subseteq A \cap (A \cup B)$ for any sets A, B.

Solution:

Let $y \in A$ be arbitrary. So certainly $y \in A$ or $y \in B$. Then by definition of union, $y \in A \cup B$. Since $y \in A$ and $y \in A \cup B$, by definition of intersection, $y \in A \cap (A \cup B)$. Since y was arbitrary, $A \subseteq A \cap (A \cup B)$.

(c) Combine part (a) and (b) to conclude that $A \cap (A \cup B) = A$ for any sets A, B.

Solution:

Since $A \cap (A \cup B) \subseteq A$ and $A \subseteq A \cap (A \cup B)$, we can deduce that $A \cap (A \cup B) = A$.

3. Subsets

Prove or disprove: for any sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Solution:

Let A, B, C be sets, and suppose $A \subseteq B$ and $B \subseteq C$. Let x be an arbitrary element of A. Then, by definition of subset, $x \in B$, and by definition of subset again, $x \in C$. Since x was an arbitrary element of A, we see that all elements of A are in C, so by definition of subset, $A \subseteq C$. So, for any sets A, B, C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

4. $\cup \to \cap$?

Prove or disprove: for all sets A and B, $A \cup B \subseteq A \cap B$.

Solution:

We wish to disprove this claim via a counterexample. Choose $A = \{1\}$, $B = \varnothing$. Note that $A \cup B = \{1\} \cup \varnothing = \{1\}$ by definition of set union. Note that $A \cap B = \{1\} \cap \varnothing = \varnothing$ by definition of set intersection. $\{1\} \not\subseteq \varnothing$, so the claim does not hold for these sets. Since we found a counterexample to the claim, we have shown that it is not the case that $A \cup B \not\subseteq A \cap B$ for all sets A and B.

5. Set Equality Proof II

We want to prove that $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

(a) First prove this with a chain of logical equivalences proof.

Solution:

Let x be arbitrary. Observe:

$$
\begin{aligned}
A \setminus (B \cap C) &\equiv (x \in A) \wedge (x \notin B \cap C) && \text{Def of Set Difference}\\
&\equiv (x \in A) \wedge \neg(x \in B \cap C) && \text{Def of element}\\
&\equiv (x \in A) \wedge \neg((x \in B) \wedge (x \in C)) && \text{Def of Intersection}\\
&\equiv (x \in A) \wedge (\neg(x \in B) \vee \neg(x \in C)) && \text{DeMorgan's Law}\\
&\equiv (x \in A) \wedge ((x \notin B) \vee (x \notin C)) && \text{Def of element}\\
&\equiv ((x \in A) \wedge (x \notin B)) \vee ((x \in A) \wedge (x \notin C)) && \text{Distributivity}\\
&\equiv (x \in A \setminus B) \vee (x \in A \setminus C) && \text{Def of Set Difference}\\
&\equiv x \in (A \setminus B) \cup (A \setminus C) && \text{Def of Union}
\end{aligned}
$$

(b) Now prove this with an English proof that is made of two subset proofs.

Solution:

Since x was arbitrary, we have shown $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

Let $x \in A \setminus (B \cap C)$ be arbitrary. Then by definition of set difference, $x \in A$ and $x \notin B \cap C$. Then by definition of intersection, $x \notin B$ or $x \notin C$. Thus (by distributive property of propositions) we have $x \in A$ and $x \notin B$, or $x \in A$ and $x \notin C$. Then by definition of set difference, $x \in (A \setminus B)$ or $x \in (A \setminus C)$. Then by definition of union, $x \in (A \setminus B) \cup (A \setminus C)$. Since x was arbitrary, we have shown $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$.

Let $x \in (A \setminus B) \cup (A \setminus C)$ be arbitrary. Then by definition of union, $x \in (A \setminus B)$ or $x \in (A \setminus C)$. Then by definition of set difference, $x \in A$ and $x \notin B$, or $x \in A$ and $x \notin C$. Then (by distributive property of propositions) $x \in A$, and $x \notin B$ or $x \notin C$. Then by definition of intersection, $x \in A$ and $x \notin (B \cap C)$. Then by definition of set difference, $x \in A \setminus (B \cap C)$. Since x was arbitrary, we have shown that $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$.

Since $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$ and $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$, we have shown $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

6. Cartesian Product Proof

Complete this English proof to show that $A \times C \subseteq (A \cup B) \times (C \cup D)$.

Let $x \in$ ____ $\times$ ____ be arbitrary.

Then x is of the form $x = (y, z)$, where $y \in$ ____ and $z \in$ ____.

Then certainly $y \in$ ____ or $y \in$ ____.

Then by definition of ________, $y \in$ (____ _ ____). (Hint: operator, set operator set)

Similarly, since $z \in$ ____, certainly $z \in$ ____ or $z \in$ ____.

Then by definition of ________, $z \in$ (____ _ ____).

Since $x = (y, z)$, then $x \in$ (____ _ ____) $\times$ (____ _ ____).

Since x was ________, we have shown ____ $\times$ ____ $\subseteq$ (____ _ ____) $\times$ (____ _ ____).

Solution:

Let $x \in A \times C$ be arbitrary. Then x is of the form $x = (y, z)$, where $y \in A$ and $z \in C$. Then certainly $y \in A$ or $y \in B$. Then by definition of union, $y \in (A \cup B)$. Similarly, since $z \in C$, certainly $z \in C$ or $z \in D$. Then by definition, $z \in (C \cup D)$. Since $x = (y, z)$, then $x \in (A \cup B) \times (C \cup D)$. Since x was arbitrary, we have shown $A \times C \subseteq (A \cup B) \times (C \cup D)$.

7. Structural Induction: Divisible by 4

Define a set $\mathfrak{B}$ of numbers by:

- 4 and 12 are in $\mathfrak{B}$
- If $x \in \mathfrak{B}$ and $y \in \mathfrak{B}$, then $x + y \in \mathfrak{B}$ and $x - y \in \mathfrak{B}$

Prove by induction that every number in $\mathfrak{B}$ is divisible by 4.

Complete the proof below:

Solution:

Let $P(b)$ be the claim that $4 \mid b$. We will prove $P(b)$ is true for all numbers $b \in \mathfrak{B}$ by structural induction.

Base Case:

- $4 \mid 4$ is trivially true, so $P(4)$ holds.
- $12 = 3 \cdot 4$, so $4 \mid 12$ and $P(12)$ holds.

Inductive Hypothesis: Suppose $P(x)$ and $P(y)$ for some arbitrary $x, y \in \mathfrak{B}$.

Inductive Step:

Goal: Prove $P(x+y)$ and $P(x-y)$

Per the IH, $4 \mid x$ and $4 \mid y$. By the definition of divides, $x = 4k$ and $y = 4j$ for some integers k, j.

Case 1: Goal: Show $P(x + y)$

$x + y = 4k + 4j = 4(k + j)$. Since integers are closed under addition, $k + j$ is an integer, so $4 \mid x + y$ and $P(x + y)$ holds.

Case 2: Goal: Show $P(x - y)$

Similarly, $x - y = 4k - 4j = 4(k - j) = 4(k + (-1 \cdot j))$. Since integers are closed under addition and multiplication, and -1 is an integer, we see that $k - j$ must be an integer. Therefore, by the definition of divides, $4 \mid x - y$ and $P(x - y)$ holds.

So, $P(t)$ holds in both cases.

Conclusion: Therefore, $P(b)$ holds for all numbers $b \in \mathfrak{B}$.

8. Structural Induction: Dictionaries

Recursive definition of a Dictionary (i.e. a Map):

- Basis Case: [] is the empty dictionary
- Recursive Case: If D is a dictionary, and a and b are elements of the universe, then $(a \to b)$ :: D is a dictionary that maps a to b (in addition to the content of D).

Recursive functions on Dictionaries:

$$\begin{aligned} \text{AllKeys}([\,]) &= [\,] & \text{len}([\,]) &= 0 \\ \text{AllKeys}((a \to b) :: \text{D}) &= a :: \text{AllKeys}(\text{D}) & \text{len}((a \to b) :: \text{D}) &= 1 + \text{len}(\text{D}) \end{aligned}$$

Recursive functions on Sets:

$$\begin{aligned} \text{len}([\,]) &= 0 \\ \text{len}(a :: \text{C}) &= 1 + \text{len}(\text{C}) \end{aligned}$$

Statement to prove:
Prove that len(D) = len(AllKeys(D)).

Solution:

Proof. Define P(D) to be len(D) = len(AllKeys(D)) for a Dictionary D. We will go by structural induction to show P(D) for all dictionaries D.
Base Case: D = []: Note that:

$$\begin{aligned} \text{len}(\text{D}) &= \text{len}([\,]) \\ &= \text{len}(\text{AllKeys}([\,])) && \text{[Definition of AllKeys]} \\ &= \text{len}(\text{AllKeys}(\text{D})) \end{aligned}$$

Inductive Hypothesis: Suppose P(C) to be true for an arbitrary dictionary C.
Inductive Step:
Let D' = $(a \to b)$ :: C. Note that:

$$\begin{aligned} \text{len}((a \to b) :: \text{C}) &= 1 + \text{len}(\text{C}) && \text{[Definition of Len]} \\ &= 1 + \text{len}(\text{AllKeys}(\text{C})) && \text{[IH]} \\ &= \text{len}(a :: \text{AllKeys}(\text{C})) && \text{[Definition of Len]} \\ &= \text{len}(\text{AllKeys}((a \to b) :: \text{C})) && \text{[Definition of AllKeys]} \end{aligned}$$

So P(D') holds.
Conclusion: Thus, the claim holds for all dictionaries D by structural induction. □

Bonus. Structural Induction: CharTrees

Recursive Definition of CharTrees:

- Basis Step: `Null` is a **CharTree**
- Recursive Step: If L, R are **CharTrees** and $c \in \Sigma$, then $\texttt{CharTree}(L, c, R)$ is also a **CharTree**

Intuitively, a **CharTree** is a tree where the non-null nodes store a `char` data element.

Recursive functions on CharTrees:

- The preorder function returns the preorder traversal of all elements in a **CharTree**.

$$\begin{aligned} &\text{preorder}(\texttt{Null}) &&= \varepsilon \\ &\text{preorder}(\texttt{CharTree}(L, c, R)) &&= c \cdot \text{preorder}(L) \cdot \text{preorder}(R) \end{aligned}$$

- The postorder function returns the postorder traversal of all elements in a **CharTree**.

$$\begin{aligned} &\text{postorder}(\texttt{Null}) &&= \varepsilon \\ &\text{postorder}(\texttt{CharTree}(L, c, R)) &&= \text{postorder}(L) \cdot \text{postorder}(R) \cdot c \end{aligned}$$

- The mirror function produces the mirror image of a **CharTree**.

$$\begin{aligned} &\text{mirror}(\texttt{Null}) &&= \texttt{Null} \\ &\text{mirror}(\texttt{CharTree}(L, c, R)) &&= \texttt{CharTree}(\text{mirror}(R), c, \text{mirror}(L)) \end{aligned}$$

- Finally, for all strings x, let the "reversal" of x (in symbols x^R) produce the string in reverse order.

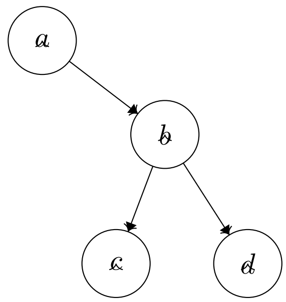
Additional Facts:
You may use the following facts:

- For any strings $x_1, ..., x_k$: $(x_1 \cdot ... \cdot x_k)^R = x_k^R \cdot ... \cdot x_1^R$
- For any character c, $c^R = c$

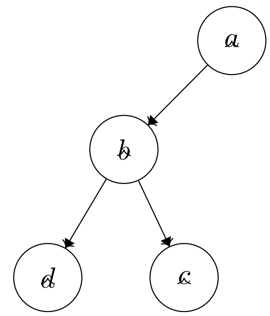
Statement to Prove:
Show that for every **CharTree** T, the reversal of the preorder traversal of T is the same as the postorder traversal of the mirror of T. In notation, you should prove that for every **CharTree**, T: $[\text{preorder}(T)]^R = \text{postorder}(\text{mirror}(T))$.

There is an example and space to work on the next page.

Example for Intuition:



Let T_i be the tree above.
preorder(T_i) =“abcd”.
T_i is built as $(\texttt{null}, a, U)$
Where U is (V, b, W),
$V = (\texttt{null}, c, \texttt{null}), W = (\texttt{null}, d, \texttt{null})$.



This tree is mirror(T_i).
postorder(mirror(T_i)) =“dcba”,
“dcba” is the reversal of “abcd” so
[preorder(T_i)]^R = postorder(mirror(T_i)) holds for T_i

Solution:

Let $P(T)$ be “[preorder(T)]^R = postorder(mirror(T))”. We show $P(T)$ holds for all **CharTree**s T by structural induction.

Base case ($T = \texttt{Null}$): preorder$(T)^R = \varepsilon^R = \varepsilon =$ postorder(Null) = postorder(mirror(Null)), so $P(\texttt{Null})$ holds.

Inductive hypothesis: Suppose $P(L) \wedge P(R)$ for arbitrary **CharTree**s L, R.

Inductive step:

We want to show $P(\texttt{CharTree}(L, c, R))$,
i.e. [preorder($\texttt{CharTree}(L, c, R)$)]^R = postorder(mirror($\texttt{CharTree}(L, c, R)$)).

Let c be an arbitrary element in Σ, and let $T = \text{CharTree}(L, c, R)$

$$
\begin{aligned}
\text{preorder}(T)^R &= [c \cdot \text{preorder}(L) \cdot \text{preorder}(R)]^R && \text{defn of preorder} \\
&= \text{preorder}(R)^R \cdot \text{preorder}(L)^R \cdot c^R && \text{Fact 1} \\
&= \text{preorder}(R)^R \cdot \text{preorder}(L)^R \cdot c && \text{Fact 2} \\
&= \text{postorder}(\text{mirror}(R)) \cdot \text{postorder}(\text{mirror}(L)) \cdot c && \text{by I.H.} \\
&= \text{postorder}(\text{CharTree}(\text{mirror}(R), c, \text{mirror}(L)) && \text{recursive defn of postorder} \\
&= \text{postorder}(\text{mirror}(\text{CharTree}(L, c, R))) && \text{recursive defn of mirror} \\
&= \text{postorder}(\text{mirror}(T)) && \text{defn of } T
\end{aligned}
$$

So $P(\texttt{CharTree}(L, c, R))$ holds.
By the principle of induction, $P(T)$ holds for all **CharTree**s T.