CSE 390Z: Mathematics for Computation Workshop

Week 6 Workshop Solutions

0. Induction: Warm-Up

Prove by induction that $5 \mid (6^n - 1)$ for all $n \in \mathbb{N}$.

Solution:

Let $P(n)$ be "$5 \mid 6^n - 1$" for all $n \in \mathbb{N}$. We will show $P(n)$ for all n by induction.

Base Case ($n = 0$). $6^0 - 1 = 1 - 1 = 0 = 0 \cdot 5$, so $5 \mid 6^0 - 1$.

Induction Hypothesis. Assume $P(k)$ holds for some arbitrary integer $k \geq 0$.

Induction Step. Goal: We aim to show $P(k+1)$, i.e. that $5 \mid (6^{k+1} - 1)$.

By the Inductive Hypothesis, we have that $5 \mid (6^k - 1)$. Then by definition of divides, $6^k - 1 = 5j$ for some $j \in \mathbb{Z}$. Observe:

$$\begin{aligned} 6^k - 1 &= 5j \\ 6^{k+1} - 6 &= 30j \\ 6^{k+1} - 1 &= 30j + 5 \\ 6^{k+1} - 1 &= 5(6j + 1) \end{aligned}$$

Since j is an integer, $j + 1$ is an integer. They by definition of divides, we have that $5 \mid (6^{k+1} - 1)$, as desired. So P$(k+1)$ holds.

Conclusion. P(n) is true for all $n \in \mathbb{N}$ by induction.

1. Induction: Equality

Prove by induction that for every $n \in \mathbb{N}$, the following equality is true:

$$0 \cdot 2^0 + 1 \cdot 2^1 + 2 \cdot 2^2 + \cdots + n \cdot 2^n = (n-1)2^{n+1} + 2.$$

Solution:

Let $P(n)$ be "$0 \cdot 2^0 + 1 \cdot 2^1 + 2 \cdot 2^2 + \cdots + n \cdot 2^n = (n-1)2^{n+1} + 2$". We will prove $P(n)$ for all $n \in \mathbb{N}$ by induction.

Base Case. $0 \cdot 2^0 = 0$ and $(0-1)2^{0+1} + 2 = -2 + 2 = 0$, so $P(0)$ is true.

Induction Hypothesis. Assume that $P(k)$ holds true for some arbitrary integer $k \geq 0$.

Induction Step We show $P(k+1)$:

$$
\begin{aligned}
&0 \cdot 2^0 + 1 \cdot 2^1 + 2 \cdot 2^2 + \cdots + (k+1) \cdot 2^{k+1} \\
&\qquad = 0 \cdot 2^0 + 1 \cdot 2^1 + 2 \cdot 2^2 + \cdots + k \cdot 2^k + (k+1) \cdot 2^{k+1} && \text{[Show another term inside "..."]} \\
&\qquad = (k-1)2^{k+1} + 2 + (k+1)2^{k+1} && \text{[Induction Hypothesis]} \\
&\qquad = ((k-1) + (k+1))2^{k+1} + 2 && \text{[Group multiples of } 2^{k+1}\text{]} \\
&\qquad = (2k)2^{k+1} + 2 && \text{[Algebra]} \\
&\qquad = k2^{k+2} + 2 && \text{[Algebra].}
\end{aligned}
$$

Therefore $P(k+1)$ holds.

Conclusion. $P(n)$ holds for all $n \in \mathbb{N}$ by induction.

2. Induction: Inequality

Prove by induction on n that for all integers $n \geq 0$ the inequality $(3+\pi)^n \geq 3^n + n\pi 3^{n-1}$ is true.

Solution:

Let $P(n)$ be "$(3+\pi)^n \geq 3^n + n\pi 3^{n-1}$". We will prove $P(n)$ is true for all $n \in \mathbb{N}$, by induction.

Base Case: **(n = 0):** $(3+\pi)^0 = 1$ and $3^0 + 0 \cdot \pi \cdot 3^{-1} = 1$, since $1 \geq 1$, $P(0)$ is true.

Inductive Hypothesis: Suppose that $P(k)$ is true for some arbitrary integer $k \in \mathbb{N}$.

Inductive Step:

$$\boxed{\text{Goal: Show } P(k+1)\text{, i.e. show } (3+\pi)^{k+1} \geq 3^{k+1} + (k+1)\pi 3^{(k+1)-1} = 3^{k+1} + (k+1)\pi 3^k}$$

$$\begin{aligned}
(3+\pi)^{k+1} &= (3+\pi)^k \cdot (3+\pi) && \text{(Factor out } (3+\pi)\text{)} \\
&\geq (3^k + k3^{k-1}\pi) \cdot (3+\pi) && \text{(By I.H., } (3+\pi) \geq 0\text{)} \\
&= 3 \cdot 3^k + 3^k\pi + 3k3^{k-1}\pi + k3^{k-1}\pi^2 && \text{(Distributive property)} \\
&= 3^{k+1} + 3^k\pi + k3^k\pi + k3^{k-1}\pi^2 && \text{(Simplify)} \\
&= 3^{k+1} + (k+1)3^k\pi + k3^{k-1}\pi^2 && \text{(Factor out } (k+1)\text{)} \\
&\geq 3^{k+1} + (k+1)\pi 3^k && (k3^{k-1}\pi^2 \geq 0)
\end{aligned}$$

Conclusion: So by induction, $P(n)$ is true for all $n \in \mathbb{N}$.

3. Inductively Odd

An 123 student learning recursion wrote a recursive Java method to determine if a number is odd or not, and needs your help proving that it is correct.

```java
public static boolean oddr(int n) {
   if (n == 0)
      return False;
   else
      return !oddr(n-1);
}
```

Help the student by writing an inductive proof to prove that for all integers $n \geq 0$, the method `oddr` returns `True` if n is an odd number, and `False` if n is not an odd number (i.e. n is even). You may recall the definitions $\text{Odd}(n) := \exists x \in \mathbb{Z}(n = 2x + 1)$ and $\text{Even}(n) := \exists x \in \mathbb{Z}(n = 2x)$; `!True = False` and `!False = True`.

Solution:

Let P(n) be "`oddr(n)` returns True if n is odd, or False if n is even". We will show that P(n) is true for all integers $n \geq 0$ by induction on n.

Base Case: (n = 0)
0 is even, so P(0) is true if `oddr(0)` returns False, which is exactly the base case of `oddr`, so P(0) is true.
Inductive Hypothesis: Suppose P(k) is true for an arbitrary integer $k \geq 0$.
Inductive Step:

- **Case 1:** $k + 1$ is even.

  If $k+1$ is even, then there is an integer x s.t. $k+1 = 2x$, so then $k = 2x-1 = 2(x-1)+1$, so therefore k is odd. We know that since $k+1 > 0$, `oddr(k+1)` should return `!oddr(k)`. By the Inductive Hypothesis, we know that since k is odd, `oddr(k)` returns True, so `oddr(k+1)` returns `!oddr(k)= False`, and $k+1$ is even, therefore P(k+1) is true.

- **Case 2:** $k + 1$ is odd.

  If $k + 1$ is odd, then there is an integer x s.t. $k + 1 = 2x + 1$, so then $k = 2x$ and therefore k is even. We know that since $k + 1 > 0$, `oddr(k+1)` should return `!oddr(k)`. By the Inductive Hypothesis, we know that since k is even, `oddr(k)` returns False, so `oddr(k+1)` returns `!oddr(k)= True`, and $k + 1$ is odd, therefore P(k+1) is true.

Then P$(k + 1)$ is true for all cases. Thus, we have shown P(n) is true for all integers $n \geq 0$ by induction.

4. Strong Induction: Stamp Collection

A store sells 3 cent and 5 cent stamps. Use strong induction to prove that you can make exactly n cents worth of stamps for all $n \geq 10$.

Hint: you'll need multiple base cases for this - think about how many steps back you need to go for your inductive step.

Solution:

Let P(n) be defined as "You can buy exactly n cents of stamps". We will prove $P(n)$ is true for all integers $n \geq 10$ by strong induction.

Base Cases: ($n = 10, 11, 12$)**:**

- $n = 10$: 10 cents of stamps can be made from two 5 cent stamps.
- $n = 11$: 11 cents of stamps can be made from one 5 cent and two 3 cent stamps.
- $n = 12$: 12 cents of stamps can be made from four 3 cent stamps.

Inductive Hypothesis: Suppose for some arbitrary integer $k \geq 12$, P(10) $\wedge$P(11) $\wedge$... $\wedge$P(k) holds.

Inductive Step:

$$\boxed{\textbf{Goal: } \text{Show } P(k+1)\text{, i.e. show that we can make } k+1 \text{ cents in stamps.}}$$

We want to buy $k + 1$ cents in stamps. By the I.H., we can buy exactly $(k + 1) - 3 = k - 2$ cents in stamps. Then, we can add another 3 cent stamp in order to buy $k + 1$ cents in stamps, so P$(k + 1)$ is true.

Note: How did we decide how many base cases to have? Well, we wanted to be able to assume P$(k-2)$, and add 3 to achieve P$(k + 1)$. Therefore we needed to be able to assume that $k - 2 \geq 10$. Adding 2 to both sides, we needed to be able to assume that $k \geq 12$. So, we have to prove the base cases up to 12, that is: $10, 11, 12$.

Another way to think about this is that we had to use a fact from 3 steps back from $k + 1$ to $k - 2$ in the IS, so we needed 3 base cases.

Conclusion: So by strong induction, P(n) is true for all integers $n \geq 10$.

5. Strong Induction: Recursively Defined Functions

Consider the function $f(n)$ defined for integers $n \geq 1$ as follows:
$f(1) = 1$ for $n = 1$
$f(2) = 4$ for $n = 2$
$f(3) = 9$ for $n = 3$
$f(n) = f(n-1) - f(n-2) + f(n-3) + 2(2n-3)$ for $n \geq 4$

Prove by strong induction that for all $n \geq 1$, $f(n) = n^2$.

Complete the induction proof below.

Solution:

1 Let P(n) be defined as " $f(n) = n^2$". We will prove $P(n)$ is true for all integers $n \geq 1$ by strong induction.

2 **Base Cases** $(n = 1, 2, 3)$**:**

- $n = 1$: $f(1) = 1 = 1^2$.
- $n = 2$: $f(2) = 4 = 2^2$.
- $n = 3$: $f(3) = 9 = 3^2$

So the base cases hold.

3 **Inductive Hypothesis:** Suppose for some arbitrary integer $k \geq 3$, P(j) is true for $1 \leq j \leq k$.

4 **Inductive Step:**

Goal: Show $P(k+1)$, i.e. show that $f(k+1) = (k+1)^2$.

$$
\begin{aligned}
f(k+1) &= f(k+1-1) - f(k+1-2) + f(k+1-3) + 2(2(k+1)-3) && \text{Definition of f} \\
&= f(k) - f(k-1) + f(k-2) + 2(2k-1) \\
&= k^2 - (k-1)^2 + (k-2)^2 + 2(2k-1) && \text{By IH} \\
&= k^2 - (k^2 - 2k + 1) + (k^2 - 4k + 4) + 4k - 2 \\
&= (k^2 - k^2 + k^2) + (2k - 4k + 4k) + (-1 + 4 - 2) \\
&= k^2 + 2k + 1 \\
&= (k+1)^2
\end{aligned}
$$

So P$(k+1)$ holds.

5 **Conclusion:** So by strong induction, P(n) is true for all integers $n \geq 1$.

6. Strong Induction: A Variation of the Stamp Problem

A store sells candy in packs of 4 and packs of 7. Let P(n) be defined as "You are able to buy n packs of candy". For example, $P(3)$ is not true, because you cannot buy exactly 3 packs of candy from the store. However, it turns out that P(n) is true for any $n \geq 18$. Use strong induction on n to prove this.

Hint: you'll need multiple base cases for this - think about how many steps back you need to go for your inductive step.

Solution:

Let P(n) be defined as "You are able to buy n packs of candy". We will prove $P(n)$ is true for all integers $n \geq 18$ by strong induction.

Base Cases: $(n = 18, 19, 20, 21)$**:**

- $n = 18$: 18 packs of candy can be made up of 2 packs of 7 and 1 pack of 4 $(18 = 2 * 7 + 1 * 4)$.
- $n = 19$: 19 packs of candy can be made up of 1 pack of 7 and 3 packs of 4 $(19 = 1 * 7 + 3 * 4)$.
- $n = 20$: 20 packs of candy can be made up of 5 packs of 4 $(20 = 5 * 4)$.
- $n = 21$: 21 packs of candy can be made up of 3 packs of 7 $(21 = 3 * 7)$.

Inductive Hypothesis: Suppose for some arbitrary integer $k \geq 21$, P(18) $\wedge$... $\wedge$P(k) hold.

Inductive Step:

$\boxed{\textbf{Goal: } \text{Show } P(k+1), \text{ i.e. show that we can buy } k+1 \text{ packs of candy.}}$

We want to buy $k+1$ packs of candy. By the I.H., we can buy exactly $k-3$ packs, so we can add another pack of 4 packs in order to buy $k+1$ packs of candy, so P$(k+1)$ is true.

Note: How did we decide how many base cases to have? Well, we wanted to be able to assume P$(k-3)$, and add 4 to achieve P$(k+1)$. Therefore we needed to be able to assume that $k - 3 \geq 18$. Adding 3 to both sides, we needed to be able to assume that $k \geq 21$. So, we have to prove the base cases up to 21, that is: $18, 19, 20, 21$.

Another way to think about this is that we had to use a fact from 4 steps back from $k+1$ to $k-3$ in the IS, so we needed 4 base cases.

Conclusion: So by strong induction, P(n) is true for all integers $n \geq 18$.