CSE 390Z: Mathematics for Computation Workshop

Week 4 Workshop Solutions

Conceptual Review

(a) Inference Rules:

Introduce $\vee$:	$\frac{A}{\therefore\ A\vee B,\ B\vee A}$	Eliminate $\vee$:	$\frac{A\vee B\ ;\ \neg A}{\therefore\ B}$
Introduce $\wedge$:	$\frac{A\ ;\ B}{\therefore\ A\wedge B}$	Eliminate $\wedge$:	$\frac{A\wedge B}{\therefore\ A\ ,\ B}$
Direct Proof:	$\frac{A\Rightarrow B}{\therefore\ A\rightarrow B}$	Modus Ponens:	$\frac{A\ ;\ A\rightarrow B}{\therefore\ B}$
Intro $\exists$:	$\frac{P(c)\text{ for some }c}{\therefore\ \exists x P(x)}$	Eliminate $\exists$:	$\frac{\exists x P(x)}{\therefore\ P(c)\text{ for a new name }c}$
Intro $\forall$:	$\frac{P(a);\ a\text{ is arbitrary}}{\therefore\ \forall x P(x)}$	Eliminate $\forall$:	$\frac{\forall x P(x)}{\therefore\ P(a);\text{ for any object }a}$

(b) What's the definition of "a divides b"?

Solution:

$a \mid b \leftrightarrow \exists k \in \mathbb{Z}\ (b = ka)$

(c) What's the Division Theorem?

Solution:

For $a \in \mathbb{Z}$, $d \in \mathbb{Z}$ with $d > 0$, there exist unique integers q, r with $0 \le r < d$, such that $a = dq + r$.

(d) How do you prove a "for all" statement in an English proof? E.g. prove $\forall x P(x)$. How do you prove a "there exists" statement? E.g. prove $\exists x P(x)$.

Solution:

To prove "for all", we show that for any **arbitrary** a in the domain, $P(a)$ holds. To prove "there exists", we show that for some specific a in the domain, $P(a)$ holds.

1. Predicate Logic Formal Proof

(a) Prove that $\forall x P(x) \to \exists x P(x)$. You may assume that the domain is nonempty.

Solution:

1.1. $\forall x P(x)$ (Assumption)

1.2. $P(a)$ (Elim $\forall$: 1.1)

1.3. $\exists x P(x)$ (Intro $\exists$: 1.2)

1. $\forall x P(x) \to \exists x P(x)$ (Direct Proof Rule, from 1.1-1.3)

(b) Given $\forall x(T(x) \to M(x))$ and $\exists x(T(x))$, prove that $\exists x(M(x))$.

Solution:

1. $\forall x(T(x) \to M(x))$ (Given)
2. $\exists x(T(x))$ (Given)
   Let r be the object that satisfies $T(r)$
3. $T(r)$ ($\exists$ elimination, from 2)
4. $T(r) \to M(r)$ ($\forall$ elimination, from 1)
5. $M(r)$ (Modus ponens, from 3 and 4)
6. $\exists x(M(x))$ ($\exists$ introduction, from 5)

(c) Given $\forall x(P(x) \to Q(x))$, prove that $(\exists x P(x)) \to (\exists y Q(y))$.

Solution:

1. $\forall x(P(x) \to Q(x))$ (Given)

2.1. $\exists x(P(x))$ (Assumption)

Let r be the object that satisfies $P(r)$

2.2. $P(r)$ ($\exists$ elimination, from 2.1)

2.3. $P(r) \to Q(r)$ ($\forall$ elimination, from 1)

2.4. $Q(r)$ (Modus Ponens, from 2.2 and 2.3)

2.5. $\exists y(Q(y))$ ($\exists$ introduction, from 2.4)

2. $(\exists x P(x)) \to (\exists y Q(y))$ (Direct Proof Rule, from 2.1-2.5)

2. More Formal Proofs: Predicate Logic!

Given $\forall x\ (P(x) \lor Q(x))$ and $\forall y\ (\neg Q(y) \lor R(y))$, prove $\exists x\ (P(x) \lor R(x))$. You may assume that the domain is not empty.

Solution:

1.	$\forall x\ (P(x) \lor Q(x))$	[Given]
2.	$\forall y\ (\neg Q(y) \lor R(y))$	[Given]
3.	$P(a) \lor Q(a)$	[Elim $\forall$: 1]
4.	$\neg Q(a) \lor R(a)$	[Elim $\forall$: 2]
5.	$Q(a) \to R(a)$	[Law of Implication: 4]
6.	$\neg\neg P(a) \lor Q(a)$	[Double Negation: 3]
7.	$\neg P(a) \to Q(a)$	[Law of Implication: 5]
	8.1. $\neg P(a)$ [Assumption]	
	8.2. $Q(a)$ [Modus Ponens: 8.1, 7]	
	8.3. $R(a)$ [Modus Ponens: 8.2, 5]	
8.	$\neg P(a) \to R(a)$	[Direct Proof]
9.	$\neg\neg P(a) \lor R(a)$	[Law of Implication: 8]
10.	$P(a) \lor R(a)$	[Double Negation: 9]
11.	$\exists x\ (P(x) \lor R(x))$	[Intro $\exists$: 10]

3. A Rational Conclusion

Note: This problem will walk you through the steps of an English proof. If you feel comfortable writing the proof already, feel free to jump directly to part (h).

Let the predicate Rational(x) be defined as $\exists a \exists b(\text{Integer}(a) \wedge \text{Integer}(b) \wedge b \neq 0 \wedge x = \frac{a}{b})$. Prove the following claim:

$$\forall x \forall y(\text{Rational}(x) \wedge \text{Rational}(y) \wedge (y \neq 0) \rightarrow \text{Rational}(\frac{x}{y}))$$

(a) Translate the claim to English.

Solution:

If x is rational and $y \neq 0$ is rational, then $\frac{x}{y}$ is rational.

(b) State the givens and declare any arbitrary variables you need to use.
Hint: there are no givens in this problem.

Solution:

Let x and y be arbitrary.

(c) State the assumptions you're making.
Hint: assume everything on the left side of the implication.

Solution:

Suppose x and y are rational numbers and that $y \neq 0$.

(d) Unroll the predicate definitions from your assumptions.

Solution:

Since x and y are rational numbers, by definition there are integers a, b, n, m with $b, n \neq 0$ such that $x = \frac{a}{b}$ and $y = \frac{m}{n}$.

(e) Manipulate what you have towards your goal (might be easier to do the next step first).

Solution:

Then $\frac{x}{y} = \frac{a/b}{m/n} = \frac{a \cdot n}{b \cdot m}$. Let $p = a \cdot n$ and $q = bm$. Note that since $y \neq 0$, m cannot be 0, and since $b \neq 0$ then $q \neq 0$. Because a, b, m, n are integers, $a \cdot n$ and $b \cdot m$ are integers.

(f) Reroll into your predicate definitions.

Solution:

Since $\frac{x}{y} = \frac{p}{q}$, p, q are integers, and $q \neq 0$, $\frac{x}{y}$ is rational.

(g) State your final claim.

Solution:

Because x and y were arbitrary, for any rational numbers x and y with $y \neq 0$, $\frac{x}{y}$ is rational.

(h) Now take these proof parts and assemble them into one cohesive English proof.

Solution:

Let x and y be arbitrary rational numbers with $y \neq 0$. Since x and y are rational numbers, by definition there are integers a, b, n, m with $b, n \neq 0$ such that $x = \frac{a}{b}$ and $y = \frac{m}{n}$. Then $\frac{x}{y} = \frac{a/b}{m/n} = \frac{a \cdot n}{b \cdot m}$. Let $p = a \cdot n$ and $q = bm$. Note that since $y \neq 0$, m cannot be 0, and since $b \neq 0$ then $q \neq 0$. Because a, b, m, n are integers, $a \cdot n$ and $b \cdot m$ are integers. Since $\frac{x}{y} = \frac{p}{q}$, p, q are integers, and $q \neq 0$, $\frac{x}{y}$ is rational. Because x and y were arbitrary, for any rational numbers x and y with $y \neq 0$ $\frac{x}{y}$ is rational.

4. Oddly Even

(a) Write a formal proof to show: If n, m are odd, then $n + m$ is even.

Let the predicates Odd(x) and Even(x) be defined as follows where the domain of discourse is integers:

$$\text{Odd}(x) := \exists y\ (x = 2y + 1)$$
$$\text{Even}(x) := \exists y\ (x = 2y)$$

Solution:

1.	Let x be an arbitrary integer.		
2.	Let y be an arbitrary integer.		
	3.1.	$\text{Odd}(x) \wedge \text{Odd}(y)$	[Assumption]
	3.2.	$\text{Odd}(x)$	[Elim $\wedge$: 3.1]
	3.3.	$\exists k\ (x = 2k + 1)$	[Definition of Odd, 3.2]
	3.4.	$x = 2k + 1$	[Elim $\exists$: 3.3]
	3.5.	$\text{Odd}(y)$	[Elim $\wedge$: 3.1]
	3.6.	$\exists k\ (y = 2k + 1)$	[Definition of Odd, 3.5]
	3.7.	$y = 2j + 1$	[Elim $\exists$: 3.7]
	3.8.	$x + y = 2k + 1 + 2j + 1$	[Algebra: 3.4, 3.7]
	3.9.	$x + y = 2(k + j + 1)$	[Algebra: 3.8]
	3.10.	$\exists r\ (x + y = 2r)$	[Intro $\exists$: 3.9]
	3.11.	$\text{Even}(x + y)$	[Definition of Even, 3.10]
3.	$\text{Odd}(x) \wedge \text{Odd}(y) \rightarrow \text{Even}(x + y)$		[Direct Proof Rule]
4.	$\forall m(\text{Odd}(x) \wedge \text{Odd}(m) \rightarrow \text{Even}(x + m))$		[Intro $\forall$: 2,3]
5.	$\forall n \forall m(\text{Odd}(n) \wedge \text{Odd}(m) \rightarrow \text{Even}(n + m))$		[Intro $\forall$: 1,4]

(b) Prove the same statement from part (a) using an English proof.

Solution:

Let n, m be arbitrary odd integers. Then by definition of odd, $n = 2k + 1$ for some integer k. Similarly by definition of odd, $m = 2j + 1$ for some integer j. Then $n + m = 2k + 1 + 2j + 1 = 2k + 2j + 2 = 2(k + j + 1)$. Then by definition, $n + m$ is even.

5. Divisibility Proof

Let the domain of discourse be integers. Consider the following claim:

$$\forall n \forall d\ ((d \mid n) \to (-d \mid n))$$

(a) Translate the claim into English.

Solution:

For integers n, d, if $d \mid n$, then $-d \mid n$.

(b) Write a formal proof to show that the claim holds.

Solution:

1. Let n be an arbitrary integer.
2. Let d be an arbitrary integer.
   - 3.1. $d \mid n$ (Assumption)
   - 3.2. $\exists k\ (n = kd)$ (Definition of divides, from 3.1)
   - 3.3. $n = jd$ ($\exists$ elimination, from 3.2)
   - 3.4. $n = (-d)(-j)$ (Algebra, from 3.3)
   - 3.5. $\exists k\ (n = k(-d))$ (Intro $\exists$, from 3.4)
   - 3.6. $-d \mid n$ (Definition of divides, from 3.5)
3. $(d \mid n) \to (-d \mid n)$ (Direct Proof Rule, from 3.1-3.6)
4. $\forall d\ ((d \mid n) \to (-d \mid n))$ (Intro $\forall$, from 3)
5. $\forall n \forall d\ ((d \mid n) \to (-d \mid n))$ (Intro $\forall$, from 4)

(c) Translate your proof to English.

Solution:

Let d, n be arbitrary integers, and suppose $d|n$. By definition of divides, there exists some integer k such that $n = dk = 1 \cdot dk$. Note that $-1 \cdot -1 = 1$. Substituting, we see $n = (-1)(-1)dk$. Rearranging, we have $n = (-d)(-1 \cdot k)$. Since k is an integer, $-1 \cdot k$ is an integer because the integers are closed under multiplication. So, by definition of divides, $-d|n$. Since d and n were arbitrary, it follows that for any integers d and n, if $d|n$, then $-d|n$.

6. Another Divisibility Proof

Write an English proof to prove that if k is an odd integer, then $4 \mid k^2 - 1$.

Solution:

Let k be an arbitrary odd integer. Then by definition of odd, $k = 2j + 1$ for some integer j. Then $k^2 - 1 = (2j+1)^2 - 1 = 4j^2 + 4j + 1 - 1 = 4j^2 + 4j = 4(j^2 + j)$. Then by definition of divides, $4 \mid k^2 - 1$.

7. Bonus: Disproving a For All Claim

Disprove the following claim:

For all integers a, b, c if $ac = bc$ then $a = b$.

Solution:

Consider $a = 1, b = 2, c = 0$. Then $ac = 0 = bc$ but $a \neq b$. This is a counterexample to the claim.

8. Bonus: Disproving an Exists Claim

Consider the following claim:

There exists an integer x such that x is even and x^2 is odd.

(a) This claim is false. Without using any formal reasoning, what does your intuition say about how to disprove this claim?

Solution:

Show that if an integer x is even, then x^2 is also even.

(b) Let the domain of discourse be integers. Define the predicates $\text{Odd}(x) := \exists k(x = 2k+1)$, and $\text{Even}(x) := \exists k(x = 2k)$. Translate the above claim to predicate logic.

Solution:

$\exists x(\text{Even}(x) \wedge \text{Odd}(x^2))$

(c) Negate the predicate logic translation. Then use a chain of logical equivalences to show that your negation is equivalent to $\forall x(\text{Even}(x) \rightarrow \text{Even}(x^2))$.

Hint: You may use the fact that $\neg\text{Odd}(a) \equiv \text{Even}(a)$.

Solution:

$$\begin{aligned}
\neg\exists x(\text{Even}(x) \wedge \text{Odd}(x^2)) &\equiv \forall x \neg(\text{Even}(x) \wedge \text{Odd}(x^2)) && \text{DeMorgan's Law for Quantifiers} \\
&\equiv \forall x(\neg\text{Even}(x) \vee \neg\text{Odd}(x^2)) && \text{DeMorgan's Law} \\
&\equiv \forall x(\neg\text{Even}(x) \vee \text{Even}(x^2)) && \text{Definition of Odd and Even} \\
&\equiv \forall x(\text{Even}(x) \rightarrow \text{Even}(x^2)) && \text{Law of Implication}
\end{aligned}$$

(d) Recall that to disprove a claim, we must prove its negation. Part (c) shows us that to disprove the above claim, we should prove that if an integer x is even, then x^2 is also even. Does this match your intuition?

Solution:

Yes!

(e) Write a proof of the fact that if an integer x is even, then x^2 is also even.

Solution:

Let x be an arbitrary integer. Suppose that x is even. Then by definition of even, there exists some integer k such that $x = 2k$.

Squaring both sides, we see that:

$$x^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2$$

Because k is an integer, then $2k^2$ is also an integer. So by definition of even, x^2 is even.

Since x was an arbitrary integer, we can conclude that for all integers x, if x is even then x^2 is even.

(f) Congrats, you have successfully disproved the claim!