CSE 390Z: Mathematics for Computation Workshop

Week 2 Workshop Problems Solutions

Conceptual Review

(a) What does it mean for two propositions to be *equivalent* ($p \equiv q$)?

Solution:

Equivalence is an assertion over all possible truth values that p and q always have the same truth values.

(b) What does it mean for two propositions to be *biconditional* ($p \leftrightarrow q$)?

Solution:

$p \leftrightarrow q$ is a proposition that may be true or false depending on the truth values of the variables p and q. $p \equiv q$ and $p \leftrightarrow q \equiv T$ have the same meaning.

(c) What are two different methods to show that two propositions are equivalent?

Solution:

The first method is to write a truth table for each proposition, and check that each row has the same truth value. The second is to use a chain of equivalences that starts at one proposition and ends at the other.

(d) What is DNF form? What is CNF form?

Solution:

DNF and CNF are two standard forms for producing a Boolean expression, given the Boolean function values. DNF is the "sum of products" form, and CNF is the "product of sums" form.

1. English to Logic Translation

Translate the English sentences below into propositional logic.

(a) Whenever I walk my dog, I make new friends.

Solution:

p: I walk my dog
q: I make new friends

$$p \rightarrow q$$

.

The promise is that we will definitely make new friends on the condition of walking our dog.

(b) I will drink coffee, if Starbucks is open or my coffeemaker works.

Solution:

p: I will drink coffee
q: Starbucks is open
r: my coffeemaker works

$$(q \vee r) \rightarrow p$$

(c) Being a U.S. citizen and over 18 is sufficient to be eligible to vote.

Solution:

p: One is a U.S. citizen
q: One is over 18
r: One is eligible to vote

$$(p \wedge q) \rightarrow r$$

The original sentence omits a subject. We introduced a dummy subject "one" to the propositions, you might have said "someone" or "a person" instead (among other options).

(d) I can go home only if I have finished my homework.

Solution:

p: I can go home.
q: I have finished my homework.

$$p \rightarrow q$$

The promise here is that if I can go home then I must have finished my homework. It can sometimes help to imagine when the sentence is broken. Is it broken if my homework is finished, but I cannot go home? No, perhaps I also have to say bye to my friends before I leave. But if I can go home with unfinished homework, then the promise is broken.

(e) Having an internet connection is necessary to log onto zoom.

Solution:

p: One has an internet connection
q: One can log onto zoom

$$q \rightarrow p$$

The internet connection is not enough (what if you don't have the meeting link?) but certainly if you are in the meeting then you have a connection.

(f) I am a student because I attend university.

Solution:

p: I am a student
q: I attend university

$$q \rightarrow p$$

This can be understood that *since* I attend university, then necessarily I am a student. You could also be a student if you do not attend a university, for example if you attend high school, but necessarily if you attend university then you must be a student.

2. Trickier Translation

For each of the following, define propositional variables and translate the sentences into logical notation.

(a) I will remember to send you the address only if you send me an e-mail message.

Solution:

$$p : \text{I will remember to send you the address}$$
$$q : \text{You send me an e-mail message}$$

$$\boxed{p \to q}$$

(b) If berries are ripe along the trail, hiking is safe if and only if grizzly bears have not been seen in the area.

Solution:

$$p : \text{Berries are ripe along the trail}$$
$$q : \text{Hiking is safe}$$
$$r : \text{Grizzly bears have not been seen in the area}$$

$$\boxed{p \to (q \leftrightarrow r)}$$

(c) Unless I am trying to type something, my cat is either eating or sleeping.

Solution:

$$p : \text{My cat is eating}$$
$$q : \text{My cat is sleeping}$$
$$r : \text{I'm trying to type}$$

$$\boxed{\neg r \to (p \oplus q)}$$

3. Implications and Vacuous Truth

Alice and Bob's teacher says in class "if a number is prime, then the number is odd." Alice and Bob both believe that the teacher is wrong, but for different reasons.

(a) Alice says "9 is odd and not prime, so the implication is false." Is Alice's justification correct? Why or why not?

Solution:

No. She gave an example where the premise (number is prime) is false, and the conclusion is true. This doesn't disprove the claim!

(b) Bob says "2 is prime and not odd, so the implication is false." Is Bob's justification correct? Why or why not?

Solution:

Yes. He gave an example where the premise (number is prime) is true, and the conclusion is false. This does disprove the claim!

(c) Recall that this is the truth table for implications. Which row does Alice's example correspond to? Which row does Bob's example correspond to?

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Solution:

Alice's example corresponds to the row where p is false and q is true. Bob's example corresponds to the row where p is true and q is false.

(d) Observe that in order to show that $p \to q$ is false, you need an example where p is true and q is false. Examples where p is false don't disprove the implication! (Nothing to write for this part).

4. DNFs and CNFs

Consider the following boolean functions $A(p, q, r)$ and $B(p, q, r)$.

p	q	r	$A(p, q, r)$	$B(p, q, r)$
T	T	T	F	T
T	T	F	F	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	T
F	F	T	F	T
F	F	F	F	F

(a) Write the DNF (ORs of ANDs) and CNF (ANDs of ORs) expressions for $A(p, q, r)$.

Solution:

DNF: $(p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge q \wedge \neg r)$

CNF: $(\neg p \vee \neg q \vee \neg r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee q \vee r) \wedge (p \vee q \vee \neg r) \wedge (p \vee q \vee r)$

(b) Write the DNF (ORs of ANDs) and CNF (ANDs of ORs) expressions for $B(p, q, r)$.

Solution:

DNF: $(p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$

CNF: $(\neg p \vee q \vee r) \wedge (p \vee \neg q \vee \neg r) \wedge (p \vee q \vee r)$

5. Boolean Algebra and Digital Circuits

Consider the following propositional logic expression:

$$\neg(\neg p \lor (p \land \neg r))$$

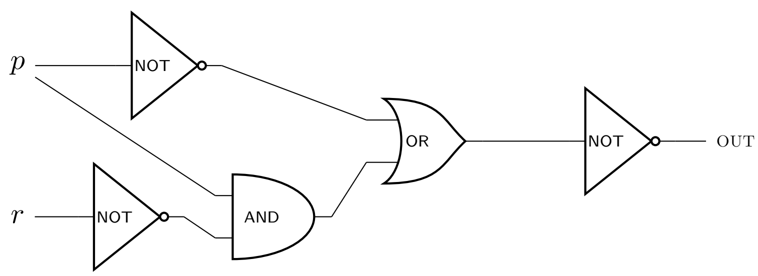
(a) Translate the proposition to Boolean Algebra notation. Do not simplify the expression.

Solution:

$$(p' + (p \cdot r'))'$$

(b) Write the proposition as a Digital Circuit.

Solution:



6. Logical Equivalences

Prove that each of the following pairs of propositional formulas are equivalent using logical equivalences.

(a) $p \to q \equiv \neg(p \land \neg q)$

Solution:

$$\begin{aligned} p \to q &\equiv \neg p \lor q && \text{Law of Implication} \\ &\equiv \neg p \lor \neg\neg q && \text{Double Negation} \\ &\equiv \neg(p \land \neg q) && \text{DeMorgan's Law} \end{aligned}$$

(b) $\neg p \to (s \to r) \equiv s \to (p \lor r)$

Solution:

$$\begin{aligned} \neg p \to (s \to r) &\equiv \neg\neg p \lor (s \to r) && \text{Law of Implication} \\ &\equiv p \lor (s \to r) && \text{Double Negation} \\ &\equiv p \lor (\neg s \lor r) && \text{Law of Implication} \\ &\equiv (p \lor \neg s) \lor r && \text{Associativity} \\ &\equiv (\neg s \lor p) \lor r && \text{Commutativity} \\ &\equiv \neg s \lor (p \lor r) && \text{Associativity} \\ &\equiv s \to (p \lor r) && \text{Law of Implication} \end{aligned}$$

(c) $\neg p \lor ((q \land p) \lor (\neg q \land p)) \equiv \mathsf{T}$

Solution:

$$
\begin{aligned}
\neg p \vee ((q \wedge p) \vee (\neg q \wedge p)) &\equiv \neg p \vee ((p \wedge q) \vee (\neg q \wedge p)) && \text{Commutativity}\\
&\equiv \neg p \vee ((p \wedge q) \vee (p \wedge \neg q)) && \text{Commutativity}\\
&\equiv \neg p \vee (p \wedge (q \vee \neg q)) && \text{Distributivity}\\
&\equiv \neg p \vee (p \wedge T) && \text{Negation}\\
&\equiv \neg p \vee p && \text{Identity}\\
&\equiv p \vee \neg p && \text{Commutativity}\\
&\equiv \mathsf{T} && \text{Negation}
\end{aligned}
$$

(d) $((p \wedge q) \rightarrow r) \equiv (p \rightarrow r) \vee (q \rightarrow r)$

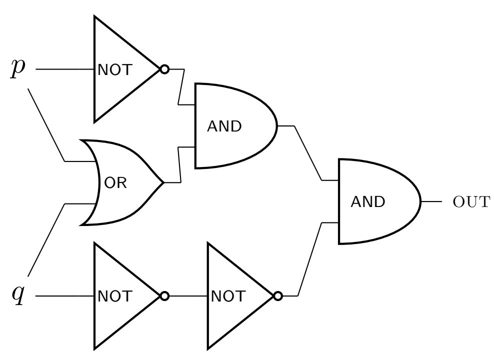
Solution:

$$
\begin{aligned}
(p \wedge q) \rightarrow r &\equiv \neg(p \wedge q) \vee r && \text{Law of Implication}\\
&\equiv (\neg p \vee \neg q) \vee r && \text{De Morgan's Law}\\
&\equiv (\neg p \vee \neg q) \vee (r \vee r) && \text{Idempotency}\\
&\equiv \neg p \vee (\neg q \vee (r \vee r)) && \text{Associativity}\\
&\equiv \neg p \vee ((\neg q \vee r) \vee r) && \text{Associativity}\\
&\equiv \neg p \vee (r \vee (\neg q \vee r)) && \text{Commutativity}\\
&\equiv \neg p \vee (r \vee (q \rightarrow r)) && \text{Law of Implication}\\
&\equiv (\neg p \vee r) \vee (q \rightarrow r) && \text{Associativity}\\
&\equiv (p \rightarrow r) \vee (q \rightarrow r) && \text{Law of Implication}
\end{aligned}
$$

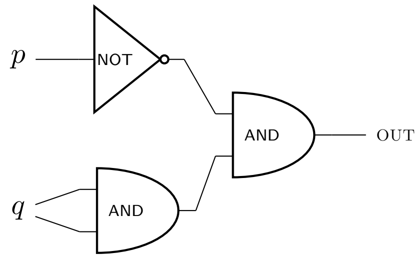
7. (Bonus) More Circuits

Convert the following ciruits into logical expressions.

(i)



(ii)



Solution:

(i) $((\neg p) \wedge (p \vee q)) \wedge \neg\neg q$

(ii) $\neg p \wedge (q \wedge q)$

8. (Bonus) XOR Truth Table

Draw a truth table for $(p \to \neg q) \to (r \oplus q)$

Solution:

p	q	r	$\neg q$	$p \to \neg q$	$r \oplus q$	$(p \to \neg q) \to (r \oplus q)$
T	T	T	F	F	F	T
T	T	F	F	F	T	T
T	F	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	T	F	F	T	T	T
F	F	T	T	T	T	T
F	F	F	T	T	F	F