CSE 390Z: Mathematics for Computation Workshop

Week 4 Workshop

Conceptual Review

(a) Inference Rules:

Introduce $\vee$:	$\frac{A}{\therefore\ A \vee B,\ B \vee A}$	Eliminate $\vee$:	$\frac{A \vee B\ ;\ \neg A}{\therefore\ B}$
Introduce $\wedge$:	$\frac{A\ ;\ B}{\therefore\ A \wedge B}$	Eliminate $\wedge$:	$\frac{A \wedge B}{\therefore\ A\ ,\ B}$
Direct Proof:	$\frac{A \Rightarrow B}{\therefore\ A \rightarrow B}$	Modus Ponens:	$\frac{A\ ;\ A \rightarrow B}{\therefore\ B}$
Intro $\exists$:	$\frac{P(c) \text{ for some } c}{\therefore\ \exists x P(x)}$	Eliminate $\exists$:	$\frac{\exists x P(x)}{\therefore\ P(c) \text{ for a new name } c}$
Intro $\forall$:	$\frac{P(a);\ a \text{ is arbitrary}}{\therefore\ \forall x P(x)}$	Eliminate $\forall$:	$\frac{\forall x P(x)}{\therefore\ P(a);\ \text{for any object } a}$

(b) What's the definition of "a divides b"?

(c) What's the Division Theorem?

(d) How do you prove a "for all" statement in an English proof? E.g. prove $\forall x P(x)$. How do you prove a "there exists" statement? E.g. prove $\exists x P(x)$.

1. Predicate Logic Formal Proof

(a) Prove that $\forall x P(x) \to \exists x P(x)$. You may assume that the domain is nonempty.

(b) Given $\forall x(T(x) \to M(x))$ and $\exists x(T(x))$, prove that $\exists x(M(x))$.

(c) Given $\forall x(P(x) \to Q(x))$, prove that $(\exists x P(x)) \to (\exists y Q(y))$.

2. More Formal Proofs: Predicate Logic!

Given $\forall x\ (P(x) \vee Q(x))$ and $\forall y\ (\neg Q(y) \vee R(y))$, prove $\exists x\ (P(x) \vee R(x))$. You may assume that the domain is not empty.

3. A Rational Conclusion

Note: This problem will walk you through the steps of an English proof. If you feel comfortable writing the proof already, feel free to jump directly to part (h).

Let the predicate Rational(x) be defined as $\exists a \exists b(\text{Integer}(a) \land \text{Integer}(b) \land b \neq 0 \land x = \frac{a}{b})$. Prove the following claim:

$$\forall x \forall y(\text{Rational}(x) \land \text{Rational}(y) \land (y \neq 0) \rightarrow \text{Rational}(\frac{x}{y}))$$

(a) Translate the claim to English.

(b) State the givens and declare any arbitrary variables you need to use.
Hint: there are no givens in this problem.

(c) State the assumptions you're making.
Hint: assume everything on the left side of the implication.

(d) Unroll the predicate definitions from your assumptions.

(e) Manipulate what you have towards your goal (might be easier to do the next step first).

(f) Reroll into your predicate definitions.

(g) State your final claim.

(h) Now take these proof parts and assemble them into one cohesive English proof.

4. Oddly Even

(a) Write a formal proof to show: If n, m are odd, then $n + m$ is even.

Let the predicates $\mathsf{Odd}(x)$ and $\mathsf{Even}(x)$ be defined as follows where the domain of discourse is integers:

$$\mathsf{Odd}(x) := \exists y\ (x = 2y + 1)$$
$$\mathsf{Even}(x) := \exists y\ (x = 2y)$$

(b) Prove the same statement from part (a) using an English proof.

5. Divisibility Proof

Let the domain of discourse be integers. Consider the following claim:

$$\forall n \forall d \; ((d \mid n) \rightarrow (-d \mid n))$$

(a) Translate the claim into English.

(b) Write a formal proof to show that the claim holds.

(c) Translate your proof to English.

6. Another Divisibility Proof

Write an English proof to prove that if k is an odd integer, then $4 \mid k^2 - 1$.

7. Bonus: Disproving a For All Claim

Disprove the following claim:

For all integers a, b, c if $ac = bc$ then $a = b$.

8. Bonus: Disproving an Exists Claim

Consider the following claim:

There exists an integer x such that x is even and x^2 is odd.

(a) This claim is false. Without using any formal reasoning, what does your intuition say about how to disprove this claim?

(b) Let the domain of discourse be integers. Define the predicates Odd$(x) := \exists k(x = 2k+1)$, and Even$(x) := \exists k(x = 2k)$. Translate the above claim to predicate logic.

(c) Negate the predicate logic translation. Then use a chain of logical equivalences to show that your negation is equivalent to $\forall x(\text{Even}(x) \rightarrow \text{Even}(x^2))$.

Hint: You may use the fact that $\neg\text{Odd}(a) \equiv \text{Even}(a)$.

(d) Recall that to disprove a claim, we must prove its negation. Part (c) shows us that to disprove the above claim, we should prove that if an integer x is even, then x^2 is also even. Does this match your intuition?

(e) Write a proof of the fact that if an integer x is even, then x^2 is also even.

(f) Congrats, you have successfully disproved the claim!