# CSE 390Z: Mathematics for Computation Workshop

## Practice 311 Midterm Solutions

Name: ______________________________

UW ID: ______________________________

**Instructions:**

- This is a **simulated practice midterm**. You will **not** be graded on your performance on this exam.
- Nevertheless, please treat this as if it is a real exam. That means that you may not discuss with your neighbors, reference outside material, or use your devices during the next 60 minute period.
- If you get stuck on a problem, consider moving on and coming back later. In the actual exam, there will likely be opportunity for partial credit.
- There are 5 problems on this exam, totaling 80 points.

**1. Predicate Translation** [15 points]
Let the domain of discourse be novels, comic books, movies, and TV shows. Translate the following statements to predicate logic, using the following predicates:

Novel$(x)$ := $x$ is a novel
Comic$(x)$ := $x$ is a comic book
Movie$(x)$ := $x$ is a movie
Show$(x)$ := $x$ is a TV show
Adaptation$(x, y)$ := $x$ is an adaptation of $y$

(a) (5 points) A novel cannot be adapted into both a movie and a TV show.

**Solution:**

$\forall x(\text{Novel}(x) \rightarrow \forall m \forall s((\text{Movie}(m) \land \text{Show}(s)) \rightarrow \neg(\text{Adaptation}(m, x) \land \text{Adaptation}(s, x)))$

(b) (5 points) Every movie is an adaptation of a novel or a comic book.

**Solution:**

$\forall m(\text{Movie}(m) \rightarrow \exists x(\text{Adaptation}(m, x) \land (\text{Novel}(x) \lor \text{Comic}(x))))$

(c) (5 points) Every novel has been adapted into exactly one movie.

**Solution:**

$\forall x(\text{Novel}(x) \rightarrow \exists m(\text{Movie}(m) \land \text{Adaptation}(m, x) \land \forall n((\text{Movie}(n) \land (n \neq m)) \rightarrow \neg\text{Adaptation}(n, x))))$

OR

$\forall x(\text{Novel}(x) \rightarrow \exists m(\text{Movie}(m) \land \text{Adaptation}(m, x) \land \forall n(\text{Adaptation}(n, x) \rightarrow (\neg\text{Movie}(n) \lor n = m))))$

OR

$\forall x(\text{Novel}(x) \rightarrow \exists m(\text{Movie}(m) \land \text{Adaptation}(m, x) \land \forall n((\text{Adaptation}(n, x) \land \text{Movie}(n)) \rightarrow (n = m))))$

*Note that a great exercise is to show that the above 3 solutions are all logically equivalent :)

**2. Boolean Algebra** [15 points]

Let $f$ be the boolean function defined as $f(x, y, z) = (x + y)' + (zy)$

(a) (5 points) Fill in the following table with the values of $f(x, y, z)$ in the last column. Feel free to use the blank columns while doing your work.

| $x$ | $y$ | $z$ | | | | $f(x, y, z)$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | | | | |
| 0 | 0 | 1 | | | | |
| 0 | 1 | 0 | | | | |
| 0 | 1 | 1 | | | | |
| 1 | 0 | 0 | | | | |
| 1 | 0 | 1 | | | | |
| 1 | 1 | 0 | | | | |
| 1 | 1 | 1 | | | | |

**Solution:**

| $x$ | $y$ | $z$ | $(x+y)$ | $(x+y)'$ | $(zy)$ | $f(x,y,z)$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 1 | 0 | 1 | 0 | 1 |
| 0 | 1 | 0 | 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 1 | 1 |

(b) (5 points) Write $f(x,y,z)$ as a Boolean algebra expression in the canonical sum-of-products form in terms of the three input bits $x, y$, and $z$.

**Solution:**

$f(x,y,z) = x'y'z' + x'y'z + x'yz + xyz$

(c) (5 points) Write $f(x,y,z)$ as a Boolean algebra expression in the canonical product-of-sums form in terms of the three input bits $x, y$, and $z$.

**Solution:**

$f(x,y,z) = (x+y'+z)(x'+y+z)(x'+y+z')(x'+y'+z)$

**3. Formal Proof with Divides** [10 points]

Prove the following statement using a formal proof:

For all integers $a, b, c$, if $a^2 \mid b$ and $b^3 \mid c$, then $a^6 \mid c$.

**Solution:**

1. Let $x, y, z$ be arbitrary integers.

| | |
|---|---|
| 2.1 $x^2 \mid y \wedge y^3 \mid z$ | Assumption |
| 2.2 $x^2 \mid y$ | Elim $\wedge$: 2.1 |
| 2.3 $y^3 \mid z$ | Elim $\wedge$: 2.2 |
| 2.4 $\exists k(y = x^2 k)$ | Definition of divides: 2.2 |
| 2.5 $\exists k(z = y^3 k)$ | Definition of divides: 2.3 |
| 2.6 $y = x^2 s$ | Elim $\exists$: 2.4 |
| 2.7 $z = y^3 t$ | Elim $\exists$: 2.5 |
| 2.8 $z = (x^2 s)^3 t = x^6(s^3 t)$ | Algebra |
| 2.9 $\exists k(z = x^6 k)$ | Intro $\exists$: 2.8 |
| 2.10 $x^6 \mid z$ | Definition of divides: 2.9 |
| 2. $x^2 \mid y \wedge y^3 \mid z \rightarrow x^6 \mid z$ | Direct Proof: 2.1 - 2.10 |
| 3. $\forall a \forall b \forall c((a^2 \mid b \wedge b^3 \mid c) \rightarrow a^6 \mid c$ | Intro $\forall$ |

**4. English Proof with Mod** [20 points]

Write an English proof to show that for all integers $n$, $n^2 \equiv_4 0$ or $n^2 \equiv_4 1$.

*Hint: You'll need to break this proof into cases based on the following definitions:*
*Even(x):=* $\exists$*k(x = 2k)*
*Odd(x):=* $\exists$*k(x = 2k + 1)*

**Solution:**
Let $n$ be an arbitrary integer.

**Case 1:** $n$ is even. Then $n = 2k$ for some integer $k$. Then $n^2 = (2k)^2 = 4k^2$. Since $k$ is an integer, $k^2$ is an integer. So $n^2$ is $4$ times an integer. Then by definition of divides, $4 \mid n^2 - 0$. Then by definition of congruence, $n^2 \equiv_4 0$.

**Case 2:** $n$ is odd. Then $n = 2k + 1$ for some integer $k$. Then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$. So $n^2 - 1 = 4(k^2 + k)$. Since $k$ is an integer, $k^2 + k$ is an integer. So $n^2 - 1$ is $4$ times an integer. Then by definition of divides, $4 \mid n^2 - 1$. Then by definition of congruence, $n^2 \equiv_4 1$.

Thus in all cases, $n^2 \equiv_4 0$ or $n^2 \equiv_4 1$. Since $n$ was arbitrary, the claim holds.

**5. Induction** [20 points]
Prove by induction that $(1+\pi)^n > 1+n\pi$ for all integers $n \geq 2$.

## Solution:

1. Let P($n$) be the statement "$(1+\pi)^n > 1+n\pi$". We prove P($n$) for all integers $n \geq 2$ by induction.

2. Base Case: When $n = 2$, the LHS is $(1+\pi)^2 = 1+2\pi+\pi^2$. The RHS is $1+2\pi$. Since $\pi^2 > 0$, $1+2\pi+\pi^2 > 1+2\pi$, so the Base Case holds.

3. Inductive Hypothesis: Suppose that P($k$) holds for some arbitrary integer $k \geq 2$. Then $(1+\pi)^k > 1+k\pi$.

4. Inductive Step:

Goal: Show $P(k+1)$, i.e. show $(1+\pi)^{k+1} > 1+(k+1)\pi$

$$
\begin{aligned}
(1+\pi)^{k+1} &= (1+\pi)(1+\pi)^k && \text{Definition of Exponent} \\
&> (1+\pi)(1+k\pi) && \text{By IH} \\
&= 1+\pi+k\pi+k\pi^2 && \text{Algebra} \\
&= 1+(k+1)\pi+k\pi^2 && \text{Algebra} \\
&> 1+(k+1)\pi && \text{Since } k\pi^2 > 0
\end{aligned}
$$

Thus $(1+\pi)^{k+1} > 1+(k+1)\pi$. So P($k+1$) holds.

5. Thus we have proven P($n$) for all integers $n \geq 2$ by induction.