CSE 390Z: Mathematics for Computation Workshop

Week 2 Workshop Solutions

Conceptual Review

(a) What is the contrapositive of $p \to q$? What is the converse of $p \to q$?

Contrapositive:

Converse:

Solution:

Contrapositive: $\neg q \to \neg p$. The important thing about the contrapositive is that it's equivalent to the original statement. That is, $p \to q \equiv \neg q \to \neg p$.
Converse: $q \to p$ The important thing about the converse is that it's not necessarily equivalent to the original statement.

(b) What is a predicate, a domain of discourse, and a quantifier?

Predicate:

Domain of Discourse:

Quantifier:

Solution:

Predicate: A function, usually based on one or more variables, that is true or false.

Domain of Discourse: The universe of values that variables come from.

Quantifier: A quantifier is a symbol that allows us to say something about how many values in the domain of discourse make some predicate true. There are two quantifiers. $\forall$ says the predicate is true for everything in the domain of discourse, and $\exists$ says the predicate is true for at least one thing in the domain of discourse.

(c) When translating to predicate logic, how do you restrict to a smaller domain in a "for all"? How do you restrict to a smaller domain in an "exists"?

Solution:

If we need to restrict something quantified by a "for all", we use **implication**. If we need to restrict something quantifies by an "exists", we use **and**.

For example, suppose the domain of discourse is all animals. We translate "all birds can fly" to $\forall x(\text{Bird}(x) \to \text{Fly}(x))$. We translate "there is a bird that can fly" to $\exists x(\text{Bird}(x) \wedge \text{Fly}(x))$.

(d) What is the difference between $\forall x \exists y (P(x, y))$ and $\exists y \forall x (P(x, y))$?

Solution:

$\forall x \exists y(P(x,y))$ means that every x has a y that satisfies $P(x,y)$. But, each x can have a different y. So, if our domain of discourse is the integers, and $P(x,y)$ is $x+y=0$, then $\forall x \exists y(P(x,y))$ means every integer x has a corresponding integer y that makes the equation $x+y=0$ true.

$\exists y \forall x(P(x,y))$ means that there is a special magical y that satisfies $P(x,y)$ for every x. Using our same $P(x,y)$, the statement $\exists y \forall x(P(x,y))$ would be false because there is no magical integer y such that $x+y=0$ for every single integer x.

Another way to think about this is $\forall$ is a like a loop, and $\exists$ is like assigning a value to a single variable. Using the same $P(x,y)$ as above, but changing our domain to integers between -1000 and 1000,

for $\forall x \exists y$, you'd have something like this

```
for (int x = -1000; x <= 1000; x++) {
    int y = -x;
    assert x + y == 0;
}
```

but for $\exists y \forall x$, you'd have something like this

```
int y = 7; // 7 was random, the point is no number would work here.
for (int x = -1000; x <= 1000; x++) {
    assert x + y == 0;
}
```

And then your code crashes and everyone is sad.

(e) What are DeMorgan's Laws for Quantifiers?

Solution:

¬∀x P(x) ≡∃x ¬P(x)
¬∃x P(x) ≡∀x ¬P(x)

1. Domains of Discourse

For the following, find a domain of discourse where the following statement is true and another where it is false. Note that for the arithmetic symbols to make sense, the domains of discourse should be sets of numbers.

(a) $\exists x(2x = 0)$

Solution:

True domain: Any set of numbers that includes 0; e.g. all natural numbers.
False domain: Any set of numbers that doesn't include 0; e.g. all positive integers.

(b) $\forall x \exists y(x + y = 0)$

Solution:

True domain: Any set of numbers that includes additive inverses; e.g. all integers.
False domain: Any set of numbers that doesn't include additive inverses; e.g. all positive integers.

(c) $\exists x \forall y(x + y = y)$

Solution:

True domain: Any set of numbers that includes 0; e.g. all natural numbers (if $x = 0$, the statement holds for all y).
False domain: Any set of numbers that doesn't include 0; e.g. all positive integers.

2. Predicate Logic Warmup

Let the domain of discourse be all animals. Let $\text{Cat}(x)$ ::= "x is a cat" and $\text{Blue}(x)$ ::= "x is blue". Translate the following statements to English.

(a) $\forall x(\text{Cat}(x) \wedge \text{Blue}(x))$

Solution:

All animals are blue cats.

(b) $\forall x(\text{Cat}(x) \rightarrow \text{Blue}(x))$

Solution:

All cats are blue.

(c) $\exists x(\text{Cat}(x) \wedge \text{Blue}(x))$

Solution:

There exists a blue cat.

Kabir translated the sentence "there exists a blue cat" to $\exists x(\text{Cat}(x) \rightarrow \text{Blue}(x))$. This is wrong! Let's understand why.

(d) Use the Law of Implications to rewrite Kabir's translation without the $\rightarrow$.

Solution:

$\exists x(\neg\text{Cat}(x) \vee \text{Blue}(x))$

(e) Translate the predicate from (d) back to English. How does this differ from the intended meaning?

Solution:

Translation: There exists an animal that is not a cat, or is blue.

The difference: If there was even one non-cat animal in the universe (e.g. a single dog), this condition would be satisfied. Similarly, if there was even one blue animal in the universe, this condition would be satisfied. So, this is a very different condition than "there exists a blue cat".

(f) This is a warning to be very careful when including an implication nested under an exists! It should almost always be avoided, unless there is a forall involved as well. (Nothing to write for this part).

3. English to Predicate Logic

Express the following sentences in predicate logic. The domain of discourse is penguins. You may use the following predicates: Love(x, y) ::= "x loves y", Dances(x) ::= "x dances", Sings(x) ::= "x sings" as well as $x = y$ and $x \neq y$

(a) There is a penguin that every penguin loves.

Solution:

$\exists x \forall y(\text{Loves}(y, x))$

(b) All penguins that sing love a penguin that does not sing.

Solution:

$\forall x(\text{Sings}(x) \to \exists y(\neg\text{Sings}(y) \land \text{Loves}(x, y)))$

(c) There is exactly one penguin that dances.

Solution:

$\exists x(\text{Dances}(x) \land \forall y((y \neq x) \to \neg\text{Dances}(y)))$

or, equivalently:

$\exists x(\text{Dances}(x) \land \forall y(\text{Dances}(y) \to (x = y)))$

(d) There exists a penguin that loves itself, but hates (does not love) every other penguin.

Solution:

$\exists x(\text{Loves}(x, x) \land \forall y((y \neq x) \to \neg\text{Loves}(x, y)))$

4. Predicate Logic to English

Translate the following sentences to English. Assume the same predicates and domain of discourse as the previous problem.

(a) $\neg\exists x(\text{Dances}(x))$

Solution:

No penguins dance.

(b) $\exists x \forall y(\text{Loves}(x, y))$

Solution:

There is a penguin that loves all penguins.

(c) $\forall x(\text{Dances}(x) \rightarrow \exists y(\text{Loves}(y, x)))$

Solution:

All penguins that dance have a penguin that loves them.

(d) $\exists x \forall y((\text{Dances}(y) \wedge \text{Sings}(y)) \rightarrow \text{Loves}(x, y))$

Solution:

There exists a penguin that loves all penguins who dance and sing.

5. Predicate Negation

Negate the predicates in question 4 (included again here for convenience)

(a) $\neg\exists x(\text{Dances}(x))$

Solution:

$$\neg\neg\exists x(\text{Dances}(x)) \equiv \exists x(\text{Dances}(x)) \qquad \text{[Double Negation]}$$

(b) $\exists x \forall y(\text{Loves}(x, y))$

Solution:

$$\begin{aligned}
\neg(\exists x \forall y(\text{Loves}(x, y))) &\equiv \forall x \neg(\forall y(\text{Loves}(x, y))) && \text{[Demorgan's Law for Quantifiers]} \\
&\equiv \forall x \exists y(\neg\text{Loves}(x, y)) && \text{[Demorgan's Law for Quantifiers]}
\end{aligned}$$

(c) $\forall x(\text{Dances}(x) \rightarrow \exists y(\text{Loves}(y, x)))$

Solution:

$$\begin{aligned}
\neg(\forall x(\text{Dances}(x) \rightarrow \exists y(\text{Loves}(y, x)))) &\equiv \exists x \neg(\text{Dances}(x) \rightarrow \exists y(\text{Loves}(y, x))) && \text{[Demorgan's Law for Quantifiers]} \\
&\equiv \exists x \neg(\neg\text{Dances}(x) \vee \exists y(\text{Loves}(y, x))) && \text{[Law of Implication]} \\
&\equiv \exists x(\neg\neg\text{Dances}(x) \wedge \neg\exists y(\text{Loves}(y, x))) && \text{[Demorgan's Law]} \\
&\equiv \exists x(\text{Dances}(x) \wedge \neg\exists y(\text{Loves}(y, x))) && \text{[Double Negation]} \\
&\equiv \exists x(\text{Dances}(x) \wedge \forall y(\neg\text{Loves}(y, x))) && \text{[Demorgan's Law for Quantifiers]}
\end{aligned}$$

(d) $\exists x \forall y((\text{Dances}(y) \wedge \text{Sings}(y)) \rightarrow \text{Loves}(x, y))$

Solution:

$$
\begin{aligned}
\neg(\exists x \forall y((\text{Dances}(y) \wedge \text{Sings}(y)) \rightarrow \text{Loves}(x, y))) &\equiv \forall x(\neg \forall y((\text{Dances}(y) \wedge \text{Sings}(y)) \rightarrow \text{Loves}(x, y))) && [\text{DM(Q)}] \\
&\equiv \forall x \exists y \neg((\text{Dances}(y) \wedge \text{Sings}(y)) \rightarrow \text{Loves}(x, y)) && [\text{DM(Q)}] \\
&\equiv \forall x \exists y \neg(\neg(\text{Dances}(y) \wedge \text{Sings}(y)) \vee \text{Loves}(x, y)) && [\text{LoI}] \\
&\equiv \forall x \exists y(\neg\neg(\text{Dances}(y) \wedge \text{Sings}(y)) \wedge \neg\text{Loves}(x, y)) && [\text{DM}] \\
&\equiv \forall x \exists y((\text{Dances}(y) \wedge \text{Sings}(y)) \wedge \neg\text{Loves}(x, y)) && [\text{DN}]
\end{aligned}
$$

6. Equivalences: Propositional Logic

Prove $((p \wedge q) \rightarrow r) \equiv (p \rightarrow r) \vee (q \rightarrow r)$ via equivalences. Note that with propositional logic, you are expected to show all steps, including commutativity and associativity.

Solution:

$$
\begin{aligned}
(p \wedge q) \rightarrow r &\equiv \neg(p \wedge q) \vee r && \text{Law of Implication} \\
&\equiv (\neg p \vee \neg q) \vee r && \text{De Morgan's Law} \\
&\equiv (\neg p \vee \neg q) \vee (r \vee r) && \text{Idempotency} \\
&\equiv \neg p \vee (\neg q \vee (r \vee r)) && \text{Associativity} \\
&\equiv \neg p \vee ((\neg q \vee r) \vee r) && \text{Associativity} \\
&\equiv \neg p \vee (r \vee (\neg q \vee r)) && \text{Commutativity} \\
&\equiv (\neg p \vee r) \vee (\neg q \vee r) && \text{Associativity} \\
&\equiv (p \rightarrow r) \vee (\neg q \vee r) && \text{Law of Implication} \\
&\equiv (p \rightarrow r) \vee (q \rightarrow r) && \text{Law of Implication}
\end{aligned}
$$

7. Boolean Algebra Equivalences

(a) Prove $p' + (p \cdot q) + (q' \cdot p) = 1$ via equivalences.

Solution:

$$
\begin{aligned}
p' + p \cdot q + q' \cdot p &\equiv p' + p \cdot q + p \cdot q' && \text{Commutativity} \\
&\equiv p' + p \cdot (q + q') && \text{Distributivity} \\
&\equiv p' + p \cdot 1 && \text{Complementarity} \\
&\equiv p' + p && \text{Identity} \\
&\equiv p + p' && \text{Commutativity} \\
&\equiv 1 && \text{Complementarity}
\end{aligned}
$$

(b) Prove $(p' + q) \cdot (q + p) = q$ via equivalences.

Solution:

$$
\begin{aligned}
(p' + q) \cdot (q + p) &\equiv (q + p') \cdot (q + p) && \text{Commutativity} \\
&\equiv q + (p' \cdot p) && \text{Distributivity} \\
&\equiv q + (p \cdot p') && \text{Commutativity} \\
&\equiv q + 0 && \text{Complementarity} \\
&\equiv q && \text{Identity}
\end{aligned}
$$

8. DNFs and CNFs

Consider the following boolean functions $A(p, q, r)$ and $B(p, q, r)$.

p	q	r	$A(p, q, r)$	$B(p, q, r)$
1	1	1	0	1
1	1	0	0	1
1	0	1	1	1
1	0	0	0	0
0	1	1	1	0
0	1	0	1	1
0	0	1	0	1
0	0	0	0	0

Recall that to write the DNF:

1. Read all the rows of the truth table where the boolean function evaluates to 1
2. Take the product of all the settings in a given 1 row
3. Take the sum of all of products from step 2

To write a CNF:

1. Read all the rows of the truth table where the boolean function evaluates to 0
2. Take the sum of the negation of all of the settings in a given 0 row
3. Take the product of all of the sums from step 2

(a) Write the DNF (sum of products) and CNF (product of sums) expressions for $A(p, q, r)$.

Solution:

DNF: $pq'r + p'qr + p'qr'$

CNF: $(p' + q' + r')(p' + q' + r)(p' + q + r)(p + q + r')(p + q + r)$

(b) Write the DNF (sum of products) and CNF (product of sums) expressions for $B(p, q, r)$.

Solution:

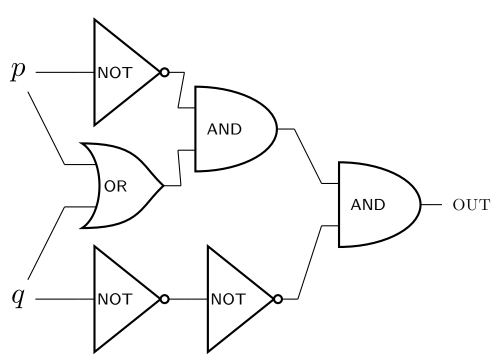
DNF: $pqr + pqr' + pq'r + p'qr' + p'q'r$

CNF: $(p' + q + r)(p + q' + r')(p + q + r)$

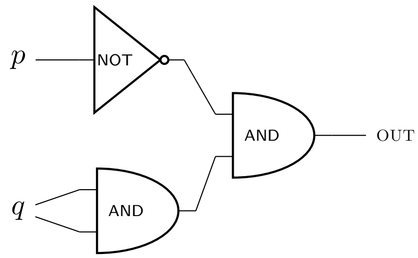
9. Circuits

Convert the following ciruits into logical expressions.

(a)



(b)



Solution:

(i) $((\neg p) \wedge (p \vee q)) \wedge \neg\neg q$

(ii) $\neg p \wedge (q \wedge q)$