CSE 390Z: Mathematics for Computation Workshop

Week 3 Workshop Problems

Conceptual Review

Predicate Logic Review

(a) When translating to predicate logic, how do you restrict to a smaller domain in a "for all"? How do you restrict to a smaller domain in an "exists"?

(b) What is the difference between $\forall x \exists y (P(x, y))$ and $\exists y \forall x (P(x, y))$?

Proof Techniques

(c) Inference Rules:

Introduce ∨:	$\frac{A}{\therefore A \vee B,\ B \vee A}$	Eliminate ∨:	$\frac{A \vee B \ ;\ \neg A}{\therefore B}$
Introduce ∧:	$\frac{A \ ;\ B}{\therefore A \wedge B}$	Eliminate ∧:	$\frac{A \wedge B}{\therefore A \ ,\ B}$
Direct Proof:	$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$	Modus Ponens:	$\frac{A \ ;\ A \rightarrow B}{\therefore B}$
Intro ∃:	$\frac{P(c) \text{ for some } c}{\therefore \exists x P(x)}$	Eliminate ∃:	$\frac{\exists x P(x)}{\therefore P(c) \text{ for a fresh } c}$
Intro ∀:	$\frac{P(a);\ a \text{ is arbitrary}}{\therefore \forall x P(x)}$	Eliminate ∀:	$\frac{\forall x P(x)}{\therefore P(a) \text{ for any } a}$

(d) Given $A \wedge B$, prove $A \vee B$

Given $P \rightarrow R$, $R \rightarrow S$, prove $P \rightarrow S$.

(e) How do we prove a "for all" statement?

1. Translations with Integers

Translate the following English sentences to predicate logic. The **domain of discourse is integers**, and you may use $=$, $\neq$, and $>$ as predicates. Assume the predicates Prime, Composite, and Even have been defined appropriately.

Note: Composite numbers are ones that have more than 2 factors.

(a) 2 is prime.

(b) Every integer greater than 1 is prime or composite, but not both.

(c) There is **exactly one** even prime.

(d) 2 is the only even prime.

(e) Some, but not all, composite integers are even.

2. Inference Proofs: Modus Ponens

(a) Prove that given $p \to q$, $\neg s \to \neg q$, and p, we can conclude s. **Hint:** You may need to use a contrapositive at some point.

(b) Prove that given $\neg\ s \to (q \vee p)$, $\neg p$, and $\neg s$, we can conclude q.

3. Inference Proofs: Direct Proof Rule

(a) Prove that given $p \to q$, we can conclude $(p \wedge r) \to q$

(b) Prove that given $p \vee q$, $q \to r$, and $r \to s$, we can conclude $\neg p \to s$.

4. Inference Proofs with Quantifiers

(a) Prove that $\forall x P(x) \rightarrow \exists x P(x)$. You may assume that the domain is nonempty.

(b) Given $\forall x(T(x) \rightarrow M(x))$ and $\exists x(T(x))$, prove that $\exists x(M(x))$.

(c) Given $\forall x(P(x) \rightarrow Q(x))$, prove that $(\exists x P(x)) \rightarrow (\exists y Q(y))$.

5. Direct Proof

Let the predicates Odd(x) and Even(x) be defined as follows where the domain of discourse is integers:

$$\text{Odd}(x) := \exists k\ (x = 2k + 1)$$

$$\text{Even}(x) := \exists k\ (x = 2k)$$

Prove the following claim using a **Direct Proof** written as an English proof (i.e., using the Direct Proof template you saw in lecture).

$$\forall x \forall y[(Even(x) \wedge Odd(y)) \rightarrow Odd(x + y)]$$

(a) Declare an arbitrary variable for each $\forall$.

(b) Assume the left side of the implication.

(c) Unroll the predicate definitions from your assumptions.

(d) Manipulate what you have towards your goal.

(e) Reroll definitions into the right side of the implication.

(f) Conclude that you have proved the claim.

(g) Now take these proof parts and assemble them into one cohesive Direct proof.

6. Oddly Even

(a) Let the predicates Odd(x) and Even(x) be defined as follows where the domain of discourse is integers:

$$\text{Odd}(x) := \exists k\ (x = 2k + 1)$$
$$\text{Even}(x) := \exists k\ (x = 2k)$$

Write an **Inference Proof** of the following claim:

$$\forall x \forall y[(Odd(x) \land Odd(y)) \to Even(x + y)]$$

Hint: You can cite the definitions of Even and Odd as well as "Algebra" to justify your steps the same way you cite Inference Rules to justify your steps.
Hint: There aren't any "Givens" here. Start by introducing two arbitrary integers a and b, then show that $(Odd(a) \land Odd(b)) \to Even(a + b)$, then use the "Intro $\forall$" rule twice.

7. Challenge: Inference Proof

Given $\forall x\ (P(x) \lor Q(x))$ and $\forall y\ (\neg Q(y) \lor R(y))$, prove $\exists x\ (P(x) \lor R(x))$. You may assume that the domain is not empty.

Hint: You can cite logical equivalence rules too.