CSE 390Z: Mathematics for Computation Workshop

Practice 311 Midterm Solutions

Name: ________________________________

UW ID: ________________________________

Instructions:

- You have eighty minutes to complete the practice exam. You will **not** be graded on your performance.
- Nevertheless, please treat this as if it is a real exam. That means that you may not discuss with your neighbors, reference outside material, or use your devices during the next 80 minute period.
- Problems are printed on both the front and back of each page!
- If you get stuck on a problem, consider moving on and coming back later. In the actual exam, there will likely be opportunity for partial credit.
- There are 4 problems on this exam.
- For multiple choice questions
  - If options are shown in ◯ circles, completely fill in the circle for the (one) best answer.
  - If options are shown in ☐ squares, completely fill in the squares for **ALL** correct answers (there may be more than one).

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1. Contradiction Number Theory

Using a **proof by contradiction**, prove the following claim:

For all integers n, if n^2 is divisible by 3 then n is divisible by 3.

(a) Negate the claim.

Solution:

Original Claim: $\forall n(3|n^2 \rightarrow 3|n)$
Negation:

$$\begin{aligned} \neg\forall n(3|n^2 \rightarrow 3|n) &\equiv \exists n\neg(3|n^2 \rightarrow 3|n) && \text{[DeMorgan's Law for Quantifiers]} \\ &\equiv \exists n\neg(\neg(3|n^2) \vee 3|n) && \text{[Law of Implication]} \\ &\equiv \exists n(3|n^2 \wedge \neg(3|n)) && \text{[DeMorgan's Law]} \end{aligned}$$

So, our final negated claim is: $\exists n(3|n^2 \wedge 3 \nmid n)$

(b) Prove the original claim using a **proof by contradiction**.
Hint: Use your solution to part (a).
Hint: You may use, without proof, that for all integers a, n with $n > 0$, $a \equiv 0 \pmod n$ iff $n|a$.

Solution:

Suppose for the sake of contradiction that there there is some integer n where n^2 is divisible by 3 and n is not divisible by 3.
Since n is not divisible by 3, by the provided hint, $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$. We consider both cases:

Case 1: ($n \equiv 1 \pmod 3$)
By definition of mod, $3|(n-1)$. By definition of divides, $n - 1 = 3k$ for some $k \in \mathbb{Z}$. Rearranging we get, $n = 3k + 1$, so $n^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$. Subtracting 1 from both sides we get $n^2 - 1 = 3(3k^2 + 2k)$. Since k is an integer, $(3k^2 + 2k)$ is also an integer, so by definition of divides, $3|n^2 - 1$. By definition of mod, $n^2 \equiv 1 \pmod 3$. By the hint given above, this means $3 \nmid n^2$, but we initially assumed n^2 was divisible by 3, so this is a contradiction.

Case 2: ($n \equiv 2 \pmod 3$)
By definition of mod, $3|(n-2)$. By definition of divides, $n - 2 = 3k$ for some $k \in \mathbb{Z}$. Rearranging we get, $n = 3k + 2$, so $n^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$. Subtracting 1 from both sides we get $n^2 - 1 = 3(3k^2 + 4k + 1)$. Since k is an integer, $(3k^2 + 4k + 1)$ is also an integer, so by definition of divides, $3|n^2 - 1$. By definition of mod, $n^2 \equiv 1 \pmod 3$. By the hint given above, this means $3 \nmid n^2$, but we initially assumed n^2 was divisible by 3, so this is a contradiction.

Since these cases are exhaustive and both result in a contradiction, we can conclude that for all integers n if n^2 is divisible by 3 then n is divisible by 3.

2. Set Theory

Write an English proof of the following claim:

For all sets A, B, C, if $C \subseteq A \cup B$ then $(C \setminus A) \subseteq B$.

Solution:

Suppose $C \subseteq A \cup B$.
Let $x \in C \setminus A$ be arbitrary.
By definition of set difference, $x \in C$ and $x \notin A$.
Since $x \in C$ and $C \subseteq A \cup B$, by definition of subset, $x \in A \cup B$.
By definition of union, $x \in A$ or $x \in B$.
Since $x \notin A$, we must have $x \in B$.
Since x was an arbitrary element of $C \setminus A$, by definition of subset, $C \setminus A \subseteq B$.
We conclude that for all sets A, B, C, if $C \subseteq A \cup B$ then $C \setminus A \subseteq B$.

Optional Alternate Solution

Explanation:

The version above was technically a direct proof, but it deviated a bit from our direct proof template. If you're curious how you would prove this following the direct proof template exactly, we've included the solution below. It's more complex and is for enrichment only.

Let's first recall the definition of subset.
$C \subseteq A \cup B$ means $\forall x(x \in C \to x \in A \cup B)$.
$C \setminus A \subseteq B$ means $\forall x(x \in C \setminus A \to x \in B)$.
Recall that when we prove something by direct proof, we assume the hypothesis (left side) of the implication is true. We then use definitions, in this case set theory definitions, to unroll the left side of the implication. Then we manipulate towards our goal. Then we use definitions to re-roll the right side (conclusion) of the implication.

This direct proof is a bit tricky because the left side and right side of our implication are themselves implications. This doesn't change our strategy. It just means that we assume the implication $\forall x(x \in C \to x \in A \cup B)$ is true, and based on that assumption, we prove that the implication $\forall x(x \in C \setminus A \to x \in B)$ must also be true.

Solution:

Suppose $C \subseteq A \cup B$.
By definition of subset $\forall x(x \in C \to x \in A \cup B)$.
Let x be arbitrary. Then, $x \in C \to x \in A \cup B$.
By the law of implication, we have $\neg(x \in C) \vee (x \in A \cup B)$.
Applying the definition of union we get $\neg(x \in C) \vee (x \in A \vee x \in B)$.
By associativity, $(\neg(x \in C) \vee x \in A) \vee x \in B$.
Using double negation we get $(\neg(x \in C) \vee \neg\neg(x \in A)) \vee x \in B$.
By DeMorgan's Law we have $\neg(x \in C \wedge \neg(x \in A)) \vee x \in B$.
Applying the definition of set difference, we have $\neg(x \in C \setminus A) \vee x \in B$.
Applying the law of implication, we get $(x \in C \setminus A) \to x \in B$.
Since x was arbitrary, we have shown that $\forall x(x \in C \setminus A \to x \in B)$.
By definition of subset, this means $C \setminus A \subseteq B$ as required.
We conclude that for all sets A, B, C, if $C \subseteq A \cup B$ then $C \setminus A \subseteq B$.

3. Induction

Prove by induction that $(1+\pi)^n > 1 + n\pi$ for all integers $n \geq 2$.

Solution:

1. Let P(n) be the statement "$(1+\pi)^n > 1 + n\pi$". We prove P(n) for all integers $n \geq 2$ by induction.

2. Base Case: When $n = 2$, the LHS is $(1+\pi)^2 = 1 + 2\pi + \pi^2$. The RHS is $1 + 2\pi$. Since $\pi^2 > 0$, $1 + 2\pi + \pi^2 > 1 + 2\pi$, so the Base Case holds.

3. Inductive Hypothesis: Suppose that P(k) holds for some arbitrary integer $k \geq 2$. That is, $(1+\pi)^k > 1 + k\pi$.

4. Inductive Step:

$$\boxed{\text{Goal: Show } P(k+1)\text{, i.e. show } (1+\pi)^{k+1} > 1 + (k+1)\pi}$$

$$\begin{aligned}
(1+\pi)^{k+1} &= (1+\pi)(1+\pi)^k && \text{Definition of Exponent} \\
&> (1+\pi)(1+k\pi) && \text{By IH} \\
&= 1 + \pi + k\pi + k\pi^2 && \text{Algebra} \\
&= 1 + (k+1)\pi + k\pi^2 && \text{Algebra} \\
&> 1 + (k+1)\pi && \text{Since } k\pi^2 > 0
\end{aligned}$$

Thus $(1+\pi)^{k+1} > 1 + (k+1)\pi$. So P($k+1$) holds.

5. Thus we have proven P(n) for all integers $n \geq 2$ by the principle of induction.

4. Multiple Choice and Short Answer

(a) You are asked to write a direct proof of the statement "For all integers x, if x is even, then x^2 is even". Which of the following assumptions should you make at the beginning of your proof.

- ◯ Assume x is even.
- ◯ Assume x is odd.
- ◯ Assume x^2 is even.
- ◯ Assume x^2 is odd.

Solution:

Option 1: Assume x is even.

(b) Which one of the following statements about modular arithmetic is true for all integers a, b, n with $n > 0$.

- ◯ If $a \equiv b \pmod{n}$, then $a^2 \equiv b^2 \pmod{2n}$
- ◯ If $a \equiv b \pmod{n}$, then $a + c \equiv b + c \pmod{n}$
- ◯ If $a \equiv b \pmod{n}$, then $a - b = n$
- ◯ If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n^2}$

Solution:

Second option: If $a \equiv b \pmod{n}$, then $a + c \equiv b + c \pmod{n}$

(c) You wish to show "If $x^2 < 1$, then $|x| < 1$" with a proof by contrapositive.

(i) What is the contrapositive of this statement?

Solution:

If $|x| \geq 1$, then $x^2 \geq 1$.

(ii) Write the first 2-3 sentences of the proof. Make sure to introduce all variables and starting assumptions.

Solution:

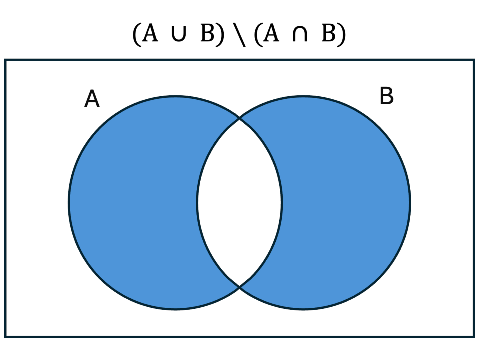
We argue by contrapositive. Let x be an arbitrary integer and suppose $|x| \geq 1$

(d) Consider the sets A and B and the set expression: $(A \cup B) \setminus (A \cap B)$

(i) Sketch or describe a Venn diagram that corresponds to this expression.

Solution:

The regions of the Venn diagram where A and B do not overlap.



(ii) Describe in words what this set represents.

Solution:

The set of all elements that are in A or B but not both.

(e) Consider the following recursive definition of a set S:
Basis Step: $5 \in S$
Recursive Step: If $x \in S$ then $x + 1 \in S$ and $x - 1 \in S$.

(i) What is the most accurate and complete description of S?

◯ The set of all natural numbers
◯ The set of all integers
◯ The set of all negative integers
◯ None of the above

Solution:

Option 2: Integers

(ii) If you were asked to use structural induction to prove that some predicate $P(x)$ was true for all $x \in S$, your inductive hypothesis would say "Suppose $P(x)$ for arbitrary $x \in S$". What would you prove in your inductive step?

Solution:

$P(x + 1)$ and $P(x - 1)$

(iii) What basis step could we use instead of **Basis Step:** $5 \in S$ without changing the set? **Select all that apply**.

☐ **Basis Step:** $0 \in S$
☐ **Basis Step:** $-5 \in S$
☐ **Basis Step:** $\pi \in S$
☐ **Basis Step:** $0 \in S, -5 \in S$

Solution:

Options 1,2,4

(f) Consider the following claim: for all integers $n \geq 12$, we can express n as $n = 4a + 5b$ for natural numbers a, b. Suppose you are asked to prove this claim by induction.

(i) What type of induction would be most appropriate?

Solution:

Strong induction

(ii) What is the minimum number of base cases would you need for this proof and what number(s) would you choose to show as your base case(s)? Note: you do not need to prove your base case(s), simply state the numbers.

Solution:

4 base cases: 12,13,14,15

(iii) Write your inductive hypothesis.

Solution:

Suppose that $P(12) \wedge P(13) \wedge ... \wedge P(k)$ are true for an arbitrary integer $k \geq 15$

(g) Which of the following are steps of weak induction? **Select all that apply.**

- ☐ Define $P(n)$. State that your proof is by induction on n.
- ☐ Base Case: Show $P(b)$ i.e. show the base case.
- ☐ Inductive Hypothesis: Suppose $P(k)$ for an arbitrary $k \geq b$.
- ☐ Inductive Step: Show $P(k+1)$ (i.e. show $P(k) \rightarrow P(k+1)$)
- ☐ Conclude by saying $P(n)$ is true for all $n \leq b$ by the principle of induction.

Solution:

ALL options EXCEPT for the last one: "Conclude by saying $P(n)$is true for all $n \leq b$ by the principle of induction."