CSE 390Z: Mathematics for Computation Workshop

Practice 311 Midterm

Name: _____

UW ID: _____

Instructions:

- You have eighty minutes to complete the practice exam. You will **not** be graded on your performance.
- Nevertheless, please treat this as if it is a real exam. That means that you may not discuss with your neighbors, reference outside material, or use your devices during the next 80 minute period.
- Problems are printed on both the front and back of each page!
- If you get stuck on a problem, consider moving on and coming back later. In the actual exam, there will likely be opportunity for partial credit.
- There are 4 problems on this exam.
- For multiple choice questions
  - If options are shown in ◯ circles, completely fill in the circle for the (one) best answer.
  - If options are shown in ☐ squares, completely fill in the squares for **ALL** correct answers (there may be more than one).

This page is intentionally left blank.

1. Contradiction Number Theory

Using a **proof by contradiction**, prove the following claim:

For all integers n, if n^2 is divisible by 3 then n is divisible by 3.

(a) Negate the claim.

(b) Prove the original claim using a **proof by contradiction**.
Hint: Use your solution to part (a).
Hint: You may use, without proof, that for all integers a, n with $n > 0$, $a \equiv 0 \pmod{n}$ iff $n|a$.

2. Set Theory

Write an English proof of the following claim:

For all sets A, B, C, if $C \subseteq A \cup B$ then $(C \setminus A) \subseteq B$.

3. Induction

Prove by induction that $(1+\pi)^n > 1 + n\pi$ for all integers $n \geq 2$.

4. Multiple Choice and Short Answer

(a) You are asked to write a direct proof of the statement "For all integers x, if x is even, then x^2 is even". Which of the following assumptions should you make at the beginning of your proof.

- ◯ Assume x is even.
- ◯ Assume x is odd.
- ◯ Assume x^2 is even.
- ◯ Assume x^2 is odd.

(b) Which one of the following statements about modular arithmetic is true for all integers a, b, n with $n > 0$.

- ◯ If $a \equiv b \pmod{n}$, then $a^2 \equiv b^2 \pmod{2n}$
- ◯ If $a \equiv b \pmod{n}$, then $a + c \equiv b + c \pmod{n}$
- ◯ If $a \equiv b \pmod{n}$, then $a - b = n$
- ◯ If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n^2}$

(c) You wish to show "If $x^2 < 1$, then $|x| < 1$" with a proof by contrapositive.

(i) What is the contrapositive of this statement?

(ii) Write the first 2-3 sentences of the proof. Make sure to introduce all variables and starting assumptions.

(d) Consider the sets A and B and the set expression: $(A \cup B) \setminus (A \cap B)$

(i) Sketch or describe a Venn diagram that corresponds to this expression.

(ii) Describe in words what this set represents.

(e) Consider the following recursive definition of a set S:
Basis Step: $5 \in S$
Recursive Step: If $x \in S$ then $x + 1 \in S$ and $x - 1 \in S$.

(i) What is the most accurate and complete description of S?

- ◯ The set of all natural numbers
- ◯ The set of all integers
- ◯ The set of all negative integers
- ◯ None of the above

(ii) If you were asked to use structural induction to prove that some predicate $P(x)$ was true for all $x \in S$, your inductive hypothesis would say "Suppose $P(x)$ for arbitrary $x \in S$". What would you prove in your inductive step?

(iii) What basis step could we use instead of **Basis Step:** $5 \in S$ without changing the set? **Select all that apply**.

- ☐ **Basis Step:** $0 \in S$
- ☐ **Basis Step:** $-5 \in S$
- ☐ **Basis Step:** $\pi \in S$
- ☐ **Basis Step:** $0 \in S, -5 \in S$

(f) Consider the following claim: for all integers $n \geq 12$, we can express n as $n = 4a + 5b$ for natural numbers a, b. Suppose you are asked to prove this claim by induction.

(i) What type of induction would be most appropriate?

(ii) What is the minimum number of base cases would you need for this proof and what number(s) would you choose to show as your base case(s)? Note: you do not need to prove your base case(s), simply state the numbers.

(iii) Write your inductive hypothesis.

(g) Which of the following are steps of weak induction? **Select all that apply.**

- ☐ Define $P(n)$. State that your proof is by induction on n.
- ☐ Base Case: Show $P(b)$ i.e. show the base case.
- ☐ Inductive Hypothesis: Suppose $P(k)$ for an arbitrary $k \geq b$.
- ☐ Inductive Step: Show $P(k+1)$ (i.e. show $P(k) \rightarrow P(k+1)$)
- ☐ Conclude by saying $P(n)$ is true for all $n \leq b$ by the principle of induction.