# CSE 390Z: Mathematics of Computing Workshop

## Week 8 Workshop Solutions

## 0. Conceptual Review

(a) Regular expression rules:
Basis: $\epsilon$, $\varnothing$, $a$ for $a \in \Sigma$
Recursive: If $A, B$ are regular expressions, $(A \cup B), AB$, and $A^*$ are regular expressions.

(b) Types of functions:
One-to-one/Injection: A function $f$ is one-to-one iff $\forall a \forall b((f(a) = f(b) \to a = b)$
Onto/Surjection: A function $f : A \to B$ is onto iff $\forall b \in B \exists a \in A(b = f(a))$
Bijection: A function $f : A \to B$ is a bijection if it is one-on-one and onto.

## 1. Regular Expressions Warmup

Consider the following Regular Expression (RegEx):

$$1(45 \cup 54)^\star 1$$

List 5 strings accepted by the RegEx and 5 strings from $T := \{1, 4, 5\}^\star$ rejected by the RegEx. Then, summarize this RegEx in your own words.

**Solution:**

**Accepted:**

- 1451
- 1541
- 145541
- 1454545451
- 11

**Rejected:**

- 1
- 1441
- 45
- 14451
- 111

This RegEx accepts exactly those strings that start and end with a $1$, and have zero or more pairs of $45$ or $54$ in the middle.

## 2. Context Free Grammars Warmup

Consider the following CFG which generates strings from the language V $:= \{0,1,2,3,4\}^*$

$$\textbf{S} \rightarrow 0\textbf{X}4$$
$$\textbf{X} \rightarrow 1\textbf{X}3 \mid 2$$

List 5 strings generated by the CFG and 5 strings from V not generated by the CFG. Then, summarize this CFG in your own words.

**Solution:**

**Accepted:**

- $024$
- $01234$
- $0112334$
- $011123334$
- $01111233334$

**Rejected:**

- $\epsilon$
- $2$
- $0244$
- $011234$
- $10234$

This CFG is all strings of the form $0\ 1^m\ 2\ 3^m\ 4$, where $m \geq 0$. That is, it's all strings made of one 0, followed by zero or more 1's, followed by a 2, followed by the same number of 3's as 1's, followed by one 4.

## 3. Simplify the RegEx

Consider the following Regular Expression (RegEx):

$$0^\star(0 \cup 1)^\star((01) \cup (11) \cup (10) \cup (00))1^\star(0 \cup 1)^\star$$

List 3 strings accepted by the RegEx and 3 strings from $S := \{0,1\}^\star$ rejected by the RegEx. Then, summarize this RegEx in your own words and write a simpler RegEx that accepts exactly the same set of strings.

**Solution:**

**Accepted:**

- 01
- 10
- 10100100101

**Rejected:**

- $\epsilon$
- 0
- 1

This RegEx accepts all binary strings that are 2 or more characters long. A simpler RegEx for this is $(0 \cup 1)(0 \cup 1)(0 \cup 1)^\star$.

## 4. Constructing RegExs and CFGs

For each of the following, construct a regular expression and CFG for the specified language.

(a) Strings from the language $S := \{a\}^*$ with an even number of $a$'s.

**Solution:**

$$(aa)^*$$

$$\textbf{S} \to aa\textbf{S}|\varepsilon$$

(b) Strings from the language $S := \{a, b\}^*$ with an even number of $a$'s.

**Solution:**

$$b^*(b^*ab^*ab^*)^*$$

$$\textbf{S} \to bS|aSaS|\epsilon$$

(c) Strings from the language $S := \{a, b\}^*$ with odd length.

**Solution:**

$$(aa \cup ab \cup ba \cup bb)^*(a \cup b)$$

$$\begin{aligned}\textbf{S} &\to \textbf{CS}|a|b\\ \textbf{C} &\to aa\textbf{C}|ab\textbf{C}|ba\textbf{C}|bb\textbf{C}|\varepsilon\end{aligned}$$

(d) (Challenge) Strings from the language $S := \{a, b\}^*$ with an even number of $a$'s or an odd number of $b$'s.

**Solution:**

$$b^*(b^*ab^*ab^*)^* \cup (a^* \cup a^*ba^*ba^*)^*b(a^* \cup a^*ba^*ba^*)^*$$

$$\begin{aligned}\textbf{S} &\to \textbf{E}|\textbf{O}b\textbf{O}\\ \textbf{E} &\to \textbf{EE}|a\textbf{E}a|b|\varepsilon\\ \textbf{O} &\to \textbf{OO}|b\textbf{O}b|a|\varepsilon\end{aligned}$$

## 5. Structural Induction: CFGs

Consider the following CFG:

$$S \to SS \mid 0S1 \mid 1S0 \mid \epsilon$$

Prove that every string generated by this CFG has an equal number of 1's and 0's.

**Hint 1:** Start by converting this CFG to a recursively defined set.
**Hint 2:** You may wish to define the functions $\#_0(x), \#_1(x)$ on a string $x$.

### Solution:

First we observe that the language defined by this CFG can be represented by a recursively defined set. Define a set $S$ as follows:
**Basis Rule**: $\epsilon \in S$
**Recursive Rule**: If $x, y \in S$, then $0x1, 1x0, xy \in S$.

Now we perform structural induction on the recursively defined set. Define the functions $\#_0(t), \#_1(t)$ to be the number of 0's and 1's respectively in the string $t$.

*Proof.* For a string $t$, let P$(t)$ be defined as "$\#_0(t) = \#_1(t)$". We will prove P$(t)$ is true for all strings $t \in S$ by structural induction.

**Base Case** ($t = \epsilon$): By definition, the empty string contains no characters, so $\#_0(t) = 0 = \#_1(t)$

**Inductive Hypothesis:** Suppose P$(x)$ and P$(y)$ hold for arbitrary strings $x, y \in S$.

**Inductive Step:**
**Case 1:** Goal: show $P(0x1)$.
By the IH, $\#_0(x) = \#_1(x)$. Then observe that:

$$\#_0(0x1) = \#_0(x) + 1 = \#_1(x) + 1 = \#_1(0x1)$$

Therefore $\#_0(0x1) = \#_1(0x1)$. This proves P$(0x1)$.

**Case 2:** Goal: show $P(1x0)$
By the IH, $\#_0(x) = \#_1(x)$. Then observe that:

$$\#_0(1x0) = \#_0(x) + 1 = \#_1(x) + 1 = \#_1(1x0)$$

Therefore $\#_0(1x0) = \#_1(1x0)$. This proves P$(1x0)$.

**Case 3:** Goal: show $P(xy)$
By the IH, $\#_0(x) = \#_1(x)$ and $\#_0(y) = \#_1(y)$. Then observe that:

$$\#_0(xy) = \#_0(x) + \#_0(y) = \#_1(x) + \#_1(y) = \#_1(xy)$$

Therefore $\#_0(xy) = \#_1(xy)$. This proves P$(xy)$.

So by structural induction, P$(t)$ is true for all strings $t \in S$. □

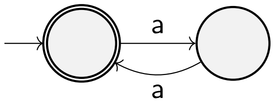
Since the recursively defined set, $S$, is exactly the set of strings generated by the CFG, we have proved that the statement is true for every string generated by the CFG too.

## 6. Constructing DFAs

For each of the following, construct a DFA for the specified language.

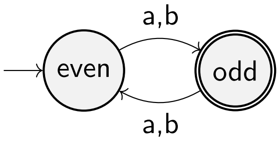
(a) Strings from the language $\Sigma := \{a\}^*$ with an even number of $a$'s.

**Solution:**



(b) Strings from the language $\Sigma = \{a, b\}$ with odd length.

**Solution:**



(c) Strings from the language $\Sigma = \{a, b\}$ with an even number of $a$'s or an odd number of $b$'s.

**Solution:**