CSE 390Z: Mathematics for Computation Workshop

Week 7 Workshop Solutions

0. Structural Induction: Divisible by 4

Define a set $\mathfrak{B}$ of numbers by:

- 4 and 12 are in $\mathfrak{B}$
- If $x \in \mathfrak{B}$ and $y \in \mathfrak{B}$, then $x + y \in \mathfrak{B}$ and $x - y \in \mathfrak{B}$

Prove by induction that every number in $\mathfrak{B}$ is divisible by 4.
Complete the proof below:

Solution:

Let $P(b)$ be the claim that $4 \mid b$. We will prove P true for all numbers $b \in \mathfrak{B}$ by structural induction.
Base Case:

- $4 \mid 4$ is trivially true, so $P(4)$ holds.
- $12 = 3 \cdot 4$, so $4 \mid 12$ and $P(12)$ holds.

Inductive Hypothesis: Suppose $P(x)$ and $P(y)$ for some arbitrary $x, y \in \mathfrak{B}$.
Inductive Step:

Goal: Prove $P(x+y)$ and $P(x-y)$

Per the IH, $4 \mid x$ and $4 \mid y$. By the definition of divides, $x = 4k$ and $y = 4j$ for some integers k, j. Then, $x + y = 4k + 4j = 4(k + j)$. Since integers are closed under addition, $k + j$ is an integer, so $4 \mid x + y$ and $P(x + y)$ holds.
Similarly, $x - y = 4k - 4j = 4(k - j) = 4(k + (-1 \cdot j))$. Since integers are closed under addition and multiplication, and -1 is an integer, we see that $k - j$ must be an integer. Therefore, by the definition of divides, $4 \mid x - y$ and $P(x - y)$ holds.
So, $P(t)$ holds in both cases.
Conclusion: Therefore, $P(b)$ holds for all numbers $b \in \mathfrak{B}$.

1. Structural Induction: a's and b's

Define a set $\mathcal{S}$ of character strings over the alphabet $\{a, b\}$ by:

- a and ab are in $\mathcal{S}$
- If $x \in \mathcal{S}$ and $y \in \mathcal{S}$, then $axb \in \mathcal{S}$ and $xy \in \mathcal{S}$

Prove by induction that every string in $\mathcal{S}$ has at least as many a's as it does b's.

Solution:

Let $P(s)$ be the claim that a string has at least many a's as it does b's. We will prove $P(s)$ true for all strings $s \in \mathcal{S}$ using structural induction.

Base Case:

- Consider $s = a$: there is one a and zero b's, so $P(a)$ holds.
- Consider $s = ab$: there is one a and one b, so $P(ab)$ holds.

Inductive Hypothesis: Suppose $P(x)$ and $P(y)$ hold for some arbitrary $x, y \in \mathcal{S}$.
Inductive Step:

Goal: Prove $P(axb)$ and $P(xy)$

First, we consider axb. We are adding one a and one b to x. Per the IH, x must have at least as many a's as it does b's. Therefore, since adding one a and one b does not change the *difference* in the number of a's and b's, axb must have at least many a's as it does b's. Thus, $P(axb)$ holds.
Second, we consider xy. Let m, n represent the number of a's in x and y respectively. Similarly, let i, j represent the number of b's in x and y. Per the IH, we know that $m \geq i$ and $n \geq j$. Adding these together, we see $m + n \geq i + j$. Therefore, xy must have at least as many a's (i.e., $m + n$ a's) as it does b's (i.e., $i + j$ b's). Thus, $P(xy)$ holds.
So, $P(t)$ holds in both cases.
Conclusion: Therefore, per the principles of structural induction, $P(s)$ holds for all strings in $\mathcal{S}$.

2. Structural Induction: CharTrees

Recursive Definition of CharTrees:

- Basis Step: `Null` is a **CharTree**
- Recursive Step: If L, R are **CharTree**s and $c \in \Sigma$, then $\texttt{CharTree}(L, c, R)$ is also a **CharTree**

Intuitively, a **CharTree** is a tree where the non-null nodes store a `char` data element.

Recursive functions on CharTrees:

- The preorder function returns the preorder traversal of all elements in a **CharTree**.

$$\begin{aligned} \text{preorder}(\texttt{Null}) &= \varepsilon \\ \text{preorder}(\texttt{CharTree}(L, c, R)) &= c \cdot \text{preorder}(L) \cdot \text{preorder}(R) \end{aligned}$$

- The postorder function returns the postorder traversal of all elements in a **CharTree**.

$$\begin{aligned} \text{postorder}(\texttt{Null}) &= \varepsilon \\ \text{postorder}(\texttt{CharTree}(L, c, R)) &= \text{postorder}(L) \cdot \text{postorder}(R) \cdot c \end{aligned}$$

- The mirror function produces the mirror image of a **CharTree**.

$$\begin{aligned} \text{mirror}(\texttt{Null}) &= \texttt{Null} \\ \text{mirror}(\texttt{CharTree}(L, c, R)) &= \texttt{CharTree}(\text{mirror}(R), c, \text{mirror}(L)) \end{aligned}$$

- Finally, for all strings x, let the "reversal" of x (in symbols x^R) produce the string in reverse order.

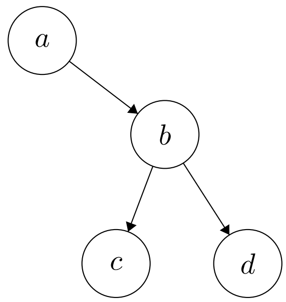
Additional Facts:
You may use the following facts:

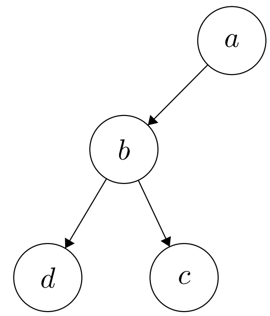
- For any strings $x_1, ..., x_k$: $(x_1 \cdot ... \cdot x_k)^R = x_k^R \cdot ... \cdot x_1^R$
- For any character c, $c^R = c$

Statement to Prove:
Show that for every **CharTree** T, the reversal of the preorder traversal of T is the same as the postorder traversal of the mirror of T. In notation, you should prove that for every **CharTree**, T: $[\text{preorder}(T)]^R = \text{postorder}(\text{mirror}(T))$.

There is an example and space to work on the next page.

Example for Intuition:





Let T_i be the tree above.
preorder(T_i) =“abcd”.
T_i is built as $(\texttt{null}, a, U)$
Where U is (V, b, W),
$V = (\texttt{null}, c, \texttt{null}), W = (\texttt{null}, d, \texttt{null})$.

This tree is mirror(T_i).
postorder(mirror(T_i)) =“dcba”,
“dcba” is the reversal of “abcd” so
$[\text{preorder}(T_i)]^R = \text{postorder}(\text{mirror}(T_i))$ holds for T_i

Solution:

Let $P(T)$ be “$[\text{preorder}(T)]^R = \text{postorder}(\text{mirror}(T))$”. We show $P(T)$ holds for all **CharTree**s T by structural induction.

Base case ($T = \texttt{Null}$): $\text{preorder}(T)^R = \varepsilon^R = \varepsilon = \text{postorder}(\texttt{Null}) = \text{postorder}(\text{mirror}(\texttt{Null}))$, so $P(\texttt{Null})$ holds.

Inductive hypothesis: Suppose $P(L) \wedge P(R)$ for arbitrary **CharTree**s L, R.

Inductive step:
We want to show $P(\texttt{CharTree}(L, c, R))$,
i.e. $[\text{preorder}(\texttt{CharTree}(L, c, R))]^R = \text{postorder}(\text{mirror}(\texttt{CharTree}(L, c, R)))$.

Let c be an arbitrary element in Σ, and let $T = \texttt{CharTree}(L, c, R)$

$$\begin{aligned}
\text{preorder}(T)^R &= [c \cdot \text{preorder}(L) \cdot \text{preorder}(R)]^R && \text{defn of preorder}\\
&= \text{preorder}(R)^R \cdot \text{preorder}(L)^R \cdot c^R && \text{Fact 1}\\
&= \text{preorder}(R)^R \cdot \text{preorder}(L)^R \cdot c && \text{Fact 2}\\
&= \text{postorder}(\text{mirror}(R)) \cdot \text{postorder}(\text{mirror}(L)) \cdot c && \text{by I.H.}\\
&= \text{postorder}(\text{CharTree}(\text{mirror}(R), c, \text{mirror}(L)) && \text{recursive defn of postorder}\\
&= \text{postorder}(\text{mirror}(\text{CharTree}(L, c, R))) && \text{recursive defn of mirror}\\
&= \text{postorder}(\text{mirror}(T)) && \text{defn of } T
\end{aligned}$$

So $P(\texttt{CharTree}(L, c, R))$ holds.
By the principle of induction, $P(T)$ holds for all **CharTree**s T.