0. Structural Induction: Divisible by 4
Define a set $\mathcal{B}$ of numbers by:
- $4$ and $12$ are in $\mathcal{B}$
- If $x \in \mathcal{B}$ and $y \in \mathcal{B}$, then $x + y \in \mathcal{B}$ and $x - y \in \mathcal{B}$

Prove by induction that every number in $\mathcal{B}$ is divisible by 4.

Complete the proof below:

Solution:
Let $P(b)$ be the claim that $4 \mid b$. We will prove $P$ true for all numbers $b \in \mathcal{B}$ by structural induction.

Base Case:
- $4 \mid 4$ is trivially true, so $P(4)$ holds.
- $12 = 3 \cdot 4$, so $4 \mid 12$ and $P(12)$ holds.

Inductive Hypothesis: Suppose $P(x)$ and $P(y)$ for some arbitrary $x, y \in \mathcal{B}$.

Inductive Step: $\boxed{\text{Goal: Prove } P(x + y) \text{ and } P(x - y)\}$
Per the IH, $4 \mid x$ and $4 \mid y$. By the definition of divides, $x = 4k$ and $y = 4j$ for some integers $k, j$. Then, $x + y = 4k + 4j = 4(k + j)$. Since integers are closed under addition, $k + j$ is an integer, so $4 \mid x + y$ and $P(x + y)$ holds.
Similarly, $x - y = 4k - 4j = 4(k - j) = 4(k + (-1 \cdot j))$. Since integers are closed under addition and multiplication, and $-1$ is an integer, we see that $k - j$ must be an integer. Therefore, by the definition of divides, $4 \mid x - y$ and $P(x - y)$ holds.
So, $P(t)$ holds in both cases.

Conclusion: Therefore, $P(b)$ holds for all numbers $b \in \mathcal{B}$.
1. **Structural Induction: a’s and b’s**

Define a set $S$ of character strings over the alphabet $\{a, b\}$ by:

- $a$ and $ab$ are in $S$
- If $x \in S$ and $y \in S$, then $axb \in S$ and $xy \in S$

Prove by induction that every string in $S$ has at least as many $a$’s as it does $b$’s.

**Solution:**

Let $P(s)$ be the claim that a string has at least many $a$’s as it does $b$’s. We will prove $P(s)$ true for all strings $s \in S$ using structural induction.

**Base Case:**

- Consider $s = a$: there is one $a$ and zero $b$’s, so $P(a)$ holds.
- Consider $s = ab$: there is one $a$ and one $b$, so $P(ab)$ holds.

**Inductive Hypothesis:** Suppose $P(x)$ and $P(y)$ hold for some arbitrary $x, y \in S$.

**Inductive Step:**

**Goal:** Prove $P(axb)$ and $P(xy)$

First, we consider $axb$. We are adding one $a$ and one $b$ to $x$. Per the IH, $x$ must have at least as many $a$’s as it does $b$’s. Therefore, since adding one $a$ and one $b$ does not change the difference in the number of $a$’s and $b$’s, $axb$ must have at least many $a$’s as it does $b$’s. Thus, $P(axb)$ holds.

Second, we consider $xy$. Let $m, n$ represent the number of $a$’s in $x$ and $y$ respectively. Similarly, let $i, j$ represent the number of $b$’s in $x$ and $y$. Per the IH, we know that $m \geq i$ and $n \geq j$. Adding these together, we see $m + n \geq i + j$. Therefore, $xy$ must have at least as many $a$’s (i.e., $m + n$ a’s) as it does $b$’s (i.e., $i + j$ b’s). Thus, $P(xy)$ holds.

So, $P(t)$ holds in both cases.

**Conclusion:** Therefore, per the principles of structural induction, $P(s)$ holds for all strings in $S$. 
2. Structural Induction: CharTrees

Recursive Definition of CharTrees:

- Basis Step: Null is a CharTree
- Recursive Step: If $L, R$ are CharTrees and $c \in \Sigma$, then CharTree($L, c, R$) is also a CharTree

Intuitively, a CharTree is a tree where the non-null nodes store a char data element.

Recursive functions on CharTrees:

- The preorder function returns the preorder traversal of all elements in a CharTree.
  \[
  \text{preorder(Null)} = \varepsilon \\
  \text{preorder(CharTree}(L, c, R)) = c \cdot \text{preorder}(L) \cdot \text{preorder}(R)
  \]

- The postorder function returns the postorder traversal of all elements in a CharTree.
  \[
  \text{postorder(Null)} = \varepsilon \\
  \text{postorder(CharTree}(L, c, R)) = \text{postorder}(L) \cdot \text{postorder}(R) \cdot c
  \]

- The mirror function produces the mirror image of a CharTree.
  \[
  \text{mirror(Null)} = \text{Null} \\
  \text{mirror(CharTree}(L, c, R)) = \text{CharTree}(\text{mirror}(R), c, \text{mirror}(L))
  \]

- Finally, for all strings $x$, let the “reversal” of $x$ (in symbols $x^R$) produce the string in reverse order.

Additional Facts:
You may use the following facts:

- For any strings $x_1, \ldots, x_k$: $(x_1 \cdot \ldots \cdot x_k)^R = x_k^R \cdot \ldots \cdot x_1^R$
- For any character $c$, $c^R = c$

Statement to Prove:
Show that for every CharTree $T$, the reversal of the preorder traversal of $T$ is the same as the postorder traversal of the mirror of $T$. In notation, you should prove that for every CharTree, $T$: $[\text{preorder}(T)]^R = \text{postorder}(\text{mirror}(T))$.

There is an example and space to work on the next page.
Example for Intuition:

Let $T_i$ be the tree above.

- $\text{preorder}(T_i) = \text{"abcd"}$. $\text{postorder}(\text{mirror}(T_i)) = \text{"dcba"}$.
- $T_i$ is built as $(\text{null}, a, U)$
  - $U$ is $(V, b, W)$
  - $V = (\text{null}, c, \text{null})$
  - $W = (\text{null}, d, \text{null})$

This tree is mirror $T_i$.

Solution:

Let $P(T)$ be $\left[\text{preorder}(T)\right]^R = \text{postorder}(\text{mirror}(T))$. We show $P(T)$ holds for all CharTrees $T$ by structural induction.

**Base case** ($T = \text{Null}$): $\text{preorder}(T)^R = \epsilon^R = \epsilon = \text{postorder}(\text{Null}) = \text{postorder}(\text{mirror}(\text{Null}))$, so $P(\text{Null})$ holds.

**Inductive hypothesis**: Suppose $P(L) \land P(R)$ for arbitrary CharTrees $L, R$.

**Inductive step**: We want to show $P(\text{CharTree}(L, c, R))$, i.e. $\left[\text{preorder}(\text{CharTree}(L, c, R))\right]^R = \text{postorder}(\text{mirror}(\text{CharTree}(L, c, R)))$.

Let $c$ be an arbitrary element in $\Sigma$, and let $T = \text{CharTree}(L, c, R)$.

\[
\text{preorder}(T)^R = \left[ c \cdot \text{preorder}(L) \cdot \text{preorder}(R) \right]^R \\
= \text{preorder}(R)^R \cdot \text{preorder}(L)^R \cdot c^R \quad \text{Fact 1} \\
= \text{preorder}(R)^R \cdot \text{preorder}(L)^R \cdot c \quad \text{Fact 2} \\
= \text{postorder}(\text{mirror}(R)) \cdot \text{postorder}(\text{mirror}(L)) \cdot c \quad \text{by I.H.} \\
= \text{postorder}(\text{CharTree}(\text{mirror}(R), c, \text{mirror}(L))) \quad \text{recursive defn of postorder} \\
= \text{postorder}(\text{mirror}(\text{CharTree}(L, c, R))) \quad \text{recursive defn of mirror} \\
= \text{postorder}(\text{mirror}(T)) \quad \text{defn of } T
\]

So $P(\text{CharTree}(L, c, R))$ holds.

By the principle of induction, $P(T)$ holds for all CharTrees $T$. 

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