CSE 390Z: Mathematics for Computation Workshop

Week 6 Workshop Solutions

Conceptual Review

Set Theory

(a) **Definitions**

Set Equality: $A = B := \forall x(x \in A \leftrightarrow x \in B)$
Subset: $A \subseteq B := \forall x(x \in A \rightarrow x \in B)$
Union: $A \cup B := \{x \ : \ x \in A \vee x \in B\}$
Intersection: $A \cap B := \{x \ : \ x \in A \wedge x \in B\}$
Set Difference: $A \setminus B = A - B := \{x \ : \ x \in A \wedge x \notin B\}$
Set Complement: $\overline{A} = A^C := \{x \ : \ x \notin A\}$
Powerset: $\mathcal{P}(A) := \{B \ : B \subseteq A\}$
Cartesian Product: $A \times B := \{(a, b) \ : a \in A, \ b \in B\}$

(b) How do we prove that for sets A and B, $A \subseteq B$?

Solution:

Let $x \in A$ be arbitrary... thus $x \in B$. Since x was arbitrary, $A \subseteq B$.

(c) How do we prove that for sets A and B, $A = B$?

Solution:

Use two subset proofs to show that $A \subseteq B$ and $B \subseteq A$.

Set Theory

1. Set Operations

Let $A = \{1, 2, 5, 6, 8\}$ and $B = \{2, 3, 5\}$.

(a) What is the set $A \cap (B \cup \{2, 8\})$?

Solution:

$\{2, 5, 8\}$

(b) What is the set $\{10\} \cup (A \setminus B)$?

Solution:

$\{1, 6, 8, 10\}$

(c) What is the set $\mathcal{P}(B)$?

Solution:

$\{\{2, 3, 5\}, \{2, 3\}, \{2, 5\}, \{3, 5\}, \{2\}, \{3\}, \{5\}, \emptyset\}$

(d) How many elements are in the set $A \times B$? List 3 of the elements.

Solution:

15 elements, for example $(1, 2), (1, 3), (1, 5)$.

2. Standard Set Proofs

(a) Prove that $A \cap B \subseteq A \cup B$ for any sets A, B.

Solution:

Let $x \in A \cap B$ be arbitrary. Then by definition of intersection, $x \in A$ and $x \in B$. So certainly $x \in A$ or $x \in B$ (using the Elim $\wedge$ and Intro $\vee$ rules). Then by definition of union, $x \in A \cup B$. Since x was arbitrary, $A \cap B \subseteq A \cup B$.

(b) Prove that $A \cap (A \cup B) = A$ for any sets A, B.

Solution:

$\Rightarrow$
Let $x \in A \cap (A \cup B)$ be arbitrary. Then by definition of intersection, $x \in A$ and $x \in A \cup B$. So, $x \in A$ must be true (Elim $\wedge$). Since x was arbitrary, $A \cap (A \cup B) \subseteq A$.

$\Leftarrow$
Let $x \in A$ be arbitrary. So certainly $x \in A$ or $x \in B$ (by the Intro $\vee$ rule). Then by definition of union, $x \in A \cup B$. Since $x \in A$ and $x \in A \cup B$, by definition of intersection, $x \in A \cap (A \cup B)$. Since x was arbitrary, $A \subseteq A \cap (A \cup B)$.

Thus we have shown that $A \cap (A \cup B) = A$ through two subset proofs.

(c) Prove that $A \cap (A \cup B) = A \cup (A \cap B)$ for any sets A, B.

Solution:

$\Rightarrow$
Let $x \in A \cap (A \cup B)$ be arbitrary. Then by definition of intersection $x \in A$ and $x \in A \cup B$. Since $x \in A$, then certainly $x \in A$ or $x \in A \cap B$ (Intro $\vee$). Then by definition of union. $x \in A \cup (A \cap B)$. Thus since x was arbitrary, we have shown $A \cap (A \cup B) \subseteq A \cup (A \cap B)$.

$\Leftarrow$
Let $x \in A \cup (A \cap B)$ be arbitrary. Then by definition of union, $x \in A$ or $x \in A \cap B$. Then by definition of intersection, $x \in A$, or $x \in A$ and $x \in B$. Then by distributivity, $x \in A$ or $x \in A$, and $x \in A$ or $x \in B$. Then by idempotency, $x \in A$, and $x \in A$ or $x \in B$. Then by definition of union, $x \in A$, and $x \in A \cup B$. Then by definition of intersection, $x \in A \cap (A \cup B)$. Thus since x was arbitrary, we have shown that $A \cup (A \cap B) \subseteq A \cap (A \cup B)$.

Thus we have shown $A \cap (A \cup B) = A \cup (A \cap B)$ through two subset proofs.

3. Cartesian Product Proof

Write an English proof to show that $A \times C \subseteq (A \cup B) \times (C \cup D)$.

Solution:

Let $x \in A \times C$ be arbitrary. Then x is of the form $x = (y, z)$, where $y \in A$ and $z \in C$. Then certainly $y \in A$ or $y \in B$ (by the Intro $\vee$ rule). Then by definition of union, $y \in (A \cup B)$. Similarly, since $z \in C$, certainly

$z \in C$ or $z \in D$. Then by definition, $z \in (C \cup D)$. Since $x = (y, z)$, then $x \in (A \cup B) \times (C \cup D)$. Since x was arbitrary, we have shown $A \times C \subseteq (A \cup B) \times (C \cup D)$.

4. Powerset Proof

Suppose that $A \subseteq B$. Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Solution:

Let X be an arbitrary set in $\mathcal{P}(A)$. By definition of power set, $X \subseteq A$. We need to show that $X \in \mathcal{P}(B)$, or equivalently, that $X \subseteq B$. Let $x \in X$ be arbitrary. Since $X \subseteq A$, it must be the case that $x \in A$. We were given that $A \subseteq B$. By definition of subset, any element of A is an element of B. So, it must also be the case that $x \in B$. Since x was arbitrary, we know any element of X is an element of B. By definition of subset, $X \subseteq B$. By definition of power set, $X \in \mathcal{P}(B)$. Since X was an arbitrary set, any set in $\mathcal{P}(A)$ is in $\mathcal{P}(B)$, or, by definition of subset, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

5. Proofs by Contradiction

For each part, write a proof by contradiction of the statement.

(a) If a is rational and ab is irrational, then b is irrational.

Solution:

Suppose for the sake of contradiction that this statement is false, meaning there exists an a, b where a is rational and ab is irrational, and b is not irrational. Then, b is rational. By definition of rational, $a = \frac{s}{t}$ and $b = \frac{x}{y}$ for some integers s, t, x, y where $t \neq 0$ and $y \neq 0$. Multiplying these together, we get $ab = \frac{sx}{ty}$. Since integers are closed under multiplication, sx, ty are integers. And since the product of two non zero integers cannot be zero, $ty \neq 0$. Thus, ab is rational. This is a contradiction since we stated that ab was irrational. Therefore, the original statement must be true.

(b) For all integers n, $4 \nmid n^2 - 3$.

Solution:

Suppose for the sake of contradiction there exists an integer n such that $4 \mid (n^2 - 3)$. Then, by definition of divides, there exists an integer k such that $n^2 - 3 = 4k$. We will consider two cases:

Case 1: n is even

By definition of even, there is some integer a where $n = 2a$. Substituting n into the equation above, we get $(2a)^2 - 3 = 4a^2 - 3 = 4k$. By algebra,

$$k = \frac{4a^2 - 3}{4} = a^2 - \frac{3}{4}$$

Since integers are closed under multiplication, a^2 must be an integer. Since $\frac{3}{4}$ is not an integer, k must not be an integer. This is a contradiction, since k was introduced as an integer.

Case 2: n is odd

By definition of odd, there is some integer b where $n = 2b + 1$, Substituting n into the equation above, we get $(2b + 1)^2 - 3 = 4b^2 + 4b + 1 - 3 = 4k$. By algebra,

$$k = \frac{4b^2 + 4b - 2}{4} = b^2 + b - \frac{1}{2}$$

Since integers are closed under multiplication and addition, $b^2 + b$ must be an integer. Since $\frac{1}{2}$ is not an integer, k is not an integer. This is a contradiction, since k was introduced as an integer.

As shown, all cases led to a contradiction, so the original statement must be true.

6. Prove the inequality

Prove by induction on n that for all $n \in$ the inequality $(3+\pi)^n \geq 3^n + n\pi 3^{n-1}$ is true.

Solution:

1. Let $P(n)$ be "$(3+\pi)^n \geq 3^n + n\pi 3^{n-1}$". We will prove $P(n)$ is true for all $n \in \mathbb{N}$, by induction.
2. **Base case** (n = 0)**:** $(3+\pi)^0 = 1$ and $3^0 + 0 \cdot \pi \cdot 3^{-1} = 1$, since $1 \geq 1$, $P(0)$ is true.
3. **Inductive Hypothesis:** Suppose that $P(k)$ is true for some arbitrary integer $k \in \mathbb{N}$.
4. **Inductive Step:**

   Goal: Show $P(k+1)$, i.e. show $(3+\pi)^{k+1} \geq 3^{k+1} + (k+1)\pi 3^{(k+1)-1} = 3^{k+1} + (k+1)\pi 3^k$

$$\begin{aligned}
(3+\pi)^{k+1} &= (3+\pi)^k \cdot (3+\pi) && \text{(Factor out } (3+\pi)\text{)} \\
&\geq (3^k + k3^{k-1}\pi) \cdot (3+\pi) && \text{(By I.H., } (3+\pi) \geq 0\text{)} \\
&= 3 \cdot 3^k + 3^k\pi + 3k3^{k-1}\pi + k3^{k-1}\pi^2 && \text{(Distributive property)} \\
&= 3^{k+1} + 3^k\pi + k3^k\pi + k3^{k-1}\pi^2 && \text{(Simplify)} \\
&= 3^{k+1} + (k+1)3^k\pi + k3^{k-1}\pi^2 && \text{(Factor out } (k+1)\text{)} \\
&\geq 3^{k+1} + (k+1)\pi 3^k && (k3^{k-1}\pi^2 \geq 0)
\end{aligned}$$

5. So by induction, $P(n)$ is true for all $n \in \mathbb{N}$.

7. Inductively Odd

An 123 student learning recursion wrote a recursive Java method to determine if a number is odd or not, and needs your help proving that it is correct.

```
public static boolean oddr(int n) {
   if (n == 0)
      return False;
   else
      return !oddr(n−1);
}
```

Help the student by writing an inductive proof to prove that for all integers $n \geq 0$, the method `oddr` returns `True` if n is an odd number, and `False` if n is not an odd number (i.e. n is even). You may recall the definitions Odd$(n) := \exists x \in \mathbb{Z}(n = 2x + 1)$ and Even$(n) := \exists x \in \mathbb{Z}(n = 2x)$; `!True = False` and `!False = True`.

Solution:

Let P(n) be "`oddr(n)` returns True if n is odd, or False if n is even". We will show that P(n) is true for all integers $n \geq 0$ by induction on n.

Base Case: (n $=$ 0)
0 is even, so P(0) is true if `oddr(0)` returns False, which is exactly the base case of `oddr`, so P(0) is true.
Inductive Hypothesis: Suppose P(k) is true for an arbitrary integer $k \geq 0$.
Inductive Step:

- **Case 1:** $k + 1$ is even.

  If $k+1$ is even, then there is an integer x s.t. $k+1 = 2x$, so then $k = 2x-1 = 2(x-1)+1$, so therefore k is odd. We know that since $k+1 > 0$, `oddr(k+1)` should return `!oddr(k)`. By the Inductive Hypothesis, we know that since k is odd, `oddr(k)` returns True, so `oddr(k+1)` returns `!oddr(k)= False`, and $k + 1$ is even, therefore P(k+1) is true.

- **Case 2:** $k + 1$ is odd.

  If $k + 1$ is odd, then there is an integer x s.t. $k + 1 = 2x + 1$, so then $k = 2x$ and therefore k is even. We know that since $k + 1 > 0$, `oddr(k+1)` should return `!oddr(k)`. By the Inductive Hypothesis, we know that since k is even, `oddr(k)` returns False, so `oddr(k+1)` returns `!oddr(k)= True`, and $k + 1$ is odd, therefore P(k+1) is true.

Then P$(k + 1)$ is true for all cases. Thus, we have shown P(n) is true for all integers $n \geq 0$ by induction.

8. Strong Induction

Consider the function $f(n)$ defined for integers $n \geq 1$ as follows:
$f(1) = 3$
$f(2) = 5$
$f(n) = 2f(n-1) - f(n-2)$
Prove using strong induction that for all $n \geq 1$, $f(n) = 2n + 1$.

Solution:

Let $P(n)$ be the claim that $f(n) = 2n + 1$. We will prove $P(n)$ for all $n \geq 1$ by strong induction.
Base case:
$f(1) = 3 = 2 * 1 + 1$
$f(2) = 5 = 2 * 2 + 1$
So $P(1)$ and $P(2)$ are both true.
Inductive Hypothesis: Suppose for some arbitrary integer $k \geq 2$, P(2) $\wedge$... $\wedge$P(k) hold.
Inductive Step:

Goal: Prove $P(k+1)$, in other words, $f(k+1) = 2(k+1) + 1$

$$
\begin{aligned}
f(k+1) &= 2f(k) - f(k-1) \\
&= 2(2(k)+1) - (2(k-1) - 1) && \text{by the IH} \\
&= 4k + 2 - (2k - 1) \\
&= 2k + 3 \\
&= 2(k+1) + 1
\end{aligned}
$$

Therefore, $f(k+1) = 2(k+1) + 1$, so $P(k+1)$ holds.
Conclusion: Therefore, $P(n)$ holds for all numbers $n \geq 1$ by strong induction.

9. Strong Induction: Collecting Candy

A store sells candy in packs of 4 and packs of 7. Let P(n) be defined as "You are able to buy n packs of candy". For example, $P(3)$ is not true, because you cannot buy exactly 3 packs of candy from the store. However, it turns out that P(n) is true for any $n \geq 18$. Use strong induction on n to prove this.

Hint: you'll need multiple base cases for this - think about how many steps back you need to go for your inductive step.

Solution:

Let P(n) be defined as "You are able to buy n packs of candy". We will prove $P(n)$ is true for all integers $n \geq 18$ by strong induction.

Base Cases: $(n = 18, 19, 20, 21)$**:**

- $n = 18$: 18 packs of candy can be made up of 2 packs of 7 and 1 pack of 4 $(18 = 2 * 7 + 1 * 4)$.
- $n = 19$: 19 packs of candy can be made up of 1 pack of 7 and 3 packs of 4 $(19 = 1 * 7 + 3 * 4)$.
- $n = 20$: 20 packs of candy can be made up of 5 packs of 4 $(20 = 5 * 4)$.
- $n = 21$: 21 packs of candy can be made up of 3 packs of 7 $(21 = 3 * 7)$.

Inductive Hypothesis: Suppose for some arbitrary integer $k \geq 21$, P(18) $\wedge$... $\wedge$P(k) hold.

Inductive Step:

Goal: Show $P(k+1)$, i.e. show that we can buy $k+1$ packs of candy.

We want to buy $k+1$ packs of candy. By the I.H., we can buy exactly $k-3$ packs, so we can add another pack of 4 packs in order to buy $k+1$ packs of candy, so P$(k+1)$ is true.

Note: How did we decide how many base cases to have? Well, we wanted to be able to assume P$(k-3)$, and add 4 to achieve P$(k+1)$. Therefore we needed to be able to assume that $k-3 \geq 18$. Adding 3 to both sides, we needed to be able to assume that $k \geq 21$. So, we have to prove the base cases up to 21, that is: $18, 19, 20, 21$.

Another way to think about this is that we had to use a fact from 4 steps back from $k+1$ to $k-3$ in the IS, so we needed 4 base cases.

Conclusion: So by strong induction, P(n) is true for all integers $n \geq 18$.