CSE 390Z: Mathematics for Computation Workshop

Practice 311 Midterm Solutions

Name: ____________________________

UW ID: ____________________________

Instructions:

- This is a **simulated practice midterm**. You will **not** be graded on your performance on this exam.
- Nevertheless, please treat this as if it is a real exam. That means that you may not discuss with your neighbors, reference outside material, or use your devices during the next 50 minute period.
- If you get stuck on a problem, consider moving on and coming back later. In the actual exam, there will likely be opportunity for partial credit.
- There are 4 problems on this exam, totaling 70 points.

1. Predicate Translation [20 points]

Let the domain of discourse be animals. Translate the following statements to predicate logic, using the following predicates:

Bird(x) := x is a bird
CanFly(x) := x can fly
Loves(x, y) := x loves y

(a) (5 points) There is a bird that loves itself.

Solution:

$\exists x(\text{Bird}(x) \land \text{Loves}(x, x))$

(b) (5 points) All flightless birds love a bird that can fly (the bird being loved can be different for each flightless bird).

Solution:

$\forall x(\neg\text{CanFly}(x) \land \text{Bird}(x) \rightarrow \exists y(\text{Bird}(y) \land \text{CanFly}(y) \land \text{Loves}(x, y))$

(c) (5 points) Every bird loves exactly one non bird.

Solution:

$\forall x(\text{Bird}(x) \rightarrow \exists y(\neg\text{Bird}(y) \land \text{Loves}(x, y) \land \forall z((z \neq y \land \neg\text{Bird}(z)) \rightarrow \neg\text{Loves}(x, z))))$

OR

$\forall x(\text{Bird}(x) \rightarrow \exists y(\neg\text{Bird}(y) \land \text{Loves}(x, y) \land \forall z(\text{Loves}(x, z) \rightarrow (\text{Bird}(z) \lor z = y))))$

OR

$\forall x(\text{Bird}(x) \rightarrow \exists y(\neg\text{Bird}(y) \land \text{Loves}(x, y) \land \forall z((\text{Loves}(x, z) \land \neg\text{Bird}(z)) \rightarrow z = y)))$

*Note that a great exercise is to show that the above 3 solutions are all logically equivalent :)

(d) (5 points) Translate the following into English:
$\exists x \forall y(\text{Bird}(x) \land (\text{CanFly}(y) \rightarrow \text{Loves}(x, y))$

Solution:

There is a bird that loves all flying animals.

2. Number Theory Proof [20 points]
Prove the following statement using an English proof:

The square of every odd integer can be written in the form $8k+1$ for some integer k.

Hint: You may want to use proof by cases for this problem.

Solution:

Let n be an arbitrary odd integer.
We want to show that n^2 can be written in the form $8k+1$ for some integer k. In other words, we want to show $n^2 = 8k+1$. Rearranging this equation, we get $n^2 - 1 = 8k$.
By definition of odd, $n = 2j+1$ for some integer j. Then, $n^2 - 1 = 4j^2 + 4j + 1 - 1 = 4(j^2 + j)$. At this point, consider two cases:

- **Case 1:** j is even.
  By definition of even, $j = 2x$ for some integer x. Then

  $$\begin{aligned} j^2 + j &= (2x)^2 + (2x) \\ &= 4x^2 + 2x \\ &= 2(2x^2 + x) \end{aligned}$$

  Plugging this into the above equation, we get

  $$n^2 - 1 = 4(2(2x^2 + x)) = 8(2x^2 + x)$$

  Let $k = 2x^2 + x$. Then

  $$n^2 = 8k + 1$$

  Since x is an integer, $2x^2 + x$ must be an integer. So, there exists an integer k such that we can write $n^2 = 8k+1$.

- **Case 2:** j is odd.
  By definition of odd, $j = 2x+1$ for some integer x. Then

  $$\begin{aligned} j^2 + j &= (2x+1)^2 + (2x+1) \\ &= 4x^2 + 4x + 1 + 2x + 1 \\ &= 4x^2 + 6x + 2 \\ &= 2(2x^2 + 3x + 1) \end{aligned}$$

  Plugging this into the above equation, we get

  $$n^2 - 1 = 4(2(2x^2 + 3x + 1)) = 8(2x^2 + 3x + 1)$$

  Let $k = 2x^2 + 3x + 1$. Then

  $$n^2 = 8k + 1$$

  Since x is an integer, $2x^2 + 3x + 1$ must be an integer. So, there exists an integer k such that we can write $n^2 = 8k+1$.

In all cases, we were able to write n^2 in the form $8k+1$. Since n was an arbitrary odd integer, this proves the original claim.

3. Induction [20 points]

Recall the Fibonacci sequence: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$.
Prove that $f_{n+1} < \left(\frac{7}{4}\right)^n$ for all $n \geq 1$.

Hint: To prove a < inequality, start with the left side of the inequality and use algebra + your inductive hypothesis to show that it is less than the right side.

Solution:

1. Let $P(n)$ be $f_{n+1} < \left(\frac{7}{4}\right)^n$. We will prove $P(n)$ true for all integers $n \geq 1$ by strong induction.

2. Base Cases
   - $n = 1$: $f_{1+1} = f_2 = f_1 + f_0 = 1 + 0 = 1 < \frac{7}{4} = \left(\frac{7}{4}\right)^1$, so $P(1)$ holds.
   - $n = 2$: $f_{2+1} = f_3 = f_2 + f_1 = 1 + 1 = 2 < \frac{49}{16} = \left(\frac{7}{4}\right)^2$, using the fact that $f_2 = 1$ from above. So, $P(2)$ holds.

3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 2$, $P(j)$ holds for all $2 \leq j \leq k$.

4. Inductive Step:

Goal: Show $P(k+1)$, i.e. show $f_{(k+1)+1} < \left(\frac{7}{4}\right)^{k+1}$

$$
\begin{aligned}
f_{(k+1)+1} &= f_{k+1} + f_k && \text{Def Fibonacci} \\
&< \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1} && \text{I.H.} \\
&= \left(\frac{7}{4}\right)^k \cdot \left[1 + \left(\frac{7}{4}\right)^{-1}\right] && \text{Factor out } \left(\frac{7}{4}\right)^k \text{ term} \\
&= \left(\frac{7}{4}\right)^k \left(\frac{11}{7}\right) && \text{Simplify fractions} \\
&= \left(\frac{7}{4}\right)^{k+1} \left(\frac{7}{4}\right)^{-1} \left(\frac{11}{7}\right) && \text{Split up } k \text{ exponent} \\
&= \left(\frac{7}{4}\right)^{k+1} \cdot \frac{44}{49} && \text{Simplify fractions} \\
&< \left(\frac{7}{4}\right)^{k+1}
\end{aligned}
$$

Thus $P(k+1)$ holds.

5. Therefore, by the principles of strong induction, the claim $P(n)$ holds for all integers $n \geq 1$.

4. Miscellaneous [10 points]

For parts a-c, write if the statement is true or false. If it is false, provide a counterexample. You do not need to provide any proofs/reasoning if the statement is true.

(a) (2 points) $4 \mid a^4$ for all integers a.

Solution:

False.
Counterexample is $a = 1$. $4 \nmid 1^4$

(b) (2 points) If $a|b$ and $b|c$, then $a|c$.

Solution:

True.
Given that $a|b$ and $b|c$, we know that $b = ax$ and $c = by$ for some integers x, y. Substituting the first equation into the second, we get $c = (ax)y = a(xy)$ which means $a|c$.

(c) (2 points) If $a \equiv b \pmod 5$ and $a \equiv 1 \pmod 5$ then $5|b$.

Solution:

False.
Counterexample: $a = 1$ and $b = 6$. $1 \equiv 6 \pmod 5$ and $1 \equiv 1 \pmod 5$ but $5 \nmid 6$.

(d) (4 points) Which of the following are equivalent to $\neg p \vee (q \to r)$? Select all that apply.

- ☐ $\neg p \vee (q \vee r)$
- ☐ $\neg(p \wedge q) \vee r$
- ☐ $(p \to r) \vee \neg q$
- ☐ $\neg p \vee (q \wedge r)$

Solution:

The second and third options are correct.