CSE 390Z: Mathematics for Computation Workshop

Practice 311 Final Solutions

Name: ______________________________

UW ID: ______________________________

Instructions:

- This is a **simulated practice final**. You will **not** be graded on your performance on this exam.
- Nevertheless, please treat this as if it is a real exam. That means that you may not discuss with your neighbors, reference outside material, or use your devices during the next 50 minute period. (Note: On the real final, you would have 110 minutes)
- If you get stuck on a problem, consider moving on and coming back later. In the actual exam, there will likely be opportunity for partial credit.
- There are 7 problems on this exam, totaling 110 points.

1. Predicate Translation [15 points]

Let the domain of discourse be trees and places. Define the following predicates:

- $\text{Tree}(x) := x$ is a tree
- $\text{InBloom}(x) := x$ is in bloom
- $\text{CampusSpot}(x) := x$ is a spot on campus
- $\text{Crowded}(x) := x$ is crowded
- $\text{Located}(x, y) := x$ is located in y

For parts a-d, translate the sentence to predicate logic:

(a) [3 points] Not all spots on campus are crowded.

Solution:

$\neg\forall x(\text{CampusSpot}(x) \rightarrow \text{Crowded}(x))$
OR
$\exists x(\text{CampusSpot}(x) \wedge \neg\text{Crowded}(x))$

(b) [3 points] There is a tree located in every spot on campus.

Solution:

$\forall x\exists y(\text{CampusSpot}(x) \rightarrow (\text{Tree}(y) \wedge \text{Located}(y, x))$

(c) [3 points] There is more than one tree located in every spot on campus.

Solution:

$\forall x\exists y\exists z(\text{CampusSpot}(x) \rightarrow (\text{Tree}(y) \wedge \text{Located}(y, x) \wedge \text{Tree}(z) \wedge \text{Located}(z, x) \wedge y \neq z))$

(d) [3 points] For any spot on campus, if a tree is in bloom there, it will be crowded.

Solution:

$\forall x\forall y((\text{CampusSpot}(x) \wedge \text{Tree}(y) \wedge \text{Located}(y, x) \wedge \text{InBloom}(y)) \rightarrow \text{Crowded}(x))$

(e) [3 points] Translate the ***negation*** of this statement to a natural English sentence:

$$\forall x\forall y(\text{Tree}(x) \wedge \text{CampusSpot}(y) \wedge \text{Located}(x, y) \rightarrow \neg\text{InBloom}(x))$$

Solution:

Not all trees on campus are not in bloom.
OR
There is a tree on campus that is in bloom.

2. Sets [15 points]

For any two sets A and B, prove $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

Solution:

Let X be an arbitrary set in $\mathcal{P}(A) \cup \mathcal{P}(B)$. By definition of union, $X \in \mathcal{P}(A)$ or $X \in \mathcal{P}(B)$. By definition of power set, $X \subseteq A$ or $X \subseteq B$.

Case 1: $X \subseteq A$

Let x be an arbitrary element in X. We have $X \subseteq A$. Thus, by definition of subset $x \in A$. So certainly, $x \in A$ or $x \in B$, and by definition of union, $x \in A \cup B$. Since x was arbitrary, by definition of subset, $X \subseteq A \cup B$. By definition of power set, $X \in \mathcal{P}(A \cup B)$.

Case 2: $X \subseteq B$

Similarly, let x be an arbitrary element in X. We have $X \subseteq B$. Thus, by definition of subset $x \in B$. So certainly, $x \in A$ or $x \in B$, and by definition of union, $x \in A \cup B$. Since x was arbitrary, by definition of subset, $X \subseteq A \cup B$. By definition of power set, $X \in \mathcal{P}(A \cup B)$.

Thus in any case $X \in \mathcal{P}(A \cup B)$.

Since X was an arbitrary set, we have proved $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

3. Induction I [15 points]
Use induction to prove the following statement:

For all positive integers n, $n^3 + 2n$ is divisible by 3

Solution:

1. Let P(n) be the statement "$3|(n^3 + 2n)$". We will prove P(n) for all integers $n \geq 1$ by induction.

2. Base Case: $n = 1$
$n^3 + 2n = (1)^3 + 2(1) = 3$. Since 3|3, this proves $P(1)$.

3. Inductive Hypothesis: Suppose that P(k) holds for some arbitrary integer $k \geq 1$. Then $3|(k^3 + 2k)$.

4. Inductive Step:

Goal: Show $P(k+1)$, i.e. show $3|((k+1)^3 + 2(k+1))$

$$\begin{aligned}(k+1)^3 + 2(k+1) &= (k^2 + 2k + 1)(k+1) + 2k + 2 \\ &= k^3 + k^2 + 2k^2 + 2k + k + 1 + 2k + 2 \\ &= k^3 + 3k^2 + 5k + 3 \\ &= (k^3 + 2k) + (3k^2 + 3k + 3)\end{aligned}$$

By the inductive hypothesis, there exists an integer j such that $k^3 + 2k = 3j$. Substituting this into the last step above, we get

$$(k+1)^3 + 2(k+1) = 3j + (3k^2 + 3k + 3) = 3(j + k^2 + k + 1)$$

Since j and k are integers, $(j + k^2 + k + 1)$ is an integer. By definition of divides, $3|((k+1)^3 + 2(k+1))$. Thus, $P(k+1)$ holds.

5. Thus we have proven P(n) for all integers $n \geq 1$ by induction. This is equivalent to the original statement.

4. Induction II [20 points]

Recursive Definition of BinaryTrees:

- Basis Steps: 0 is a `BinaryTree` and 1 is a `BinaryTree`
- Recursive Step: If L, R are `BinaryTrees`, then $(L, 0, R)$ and $(L, 1, R)$ are also `BinaryTrees`

Intuitively, a `BinaryTree` is a binary tree where each node stores either a 0 or a 1.

Recursive functions on BinaryTrees:
The sum function returns the sum of all nodes in a `BinaryTrees`.

$$\begin{aligned}
\mathsf{sum}(0) &= 0 \\
\mathsf{sum}(1) &= 1 \\
\mathsf{sum}((L,0,R)) &= \mathsf{sum}(L) + \mathsf{sum}(R) \\
\mathsf{sum}((L,1,R)) &= \mathsf{sum}(L) + \mathsf{sum}(R) + 1
\end{aligned}$$

Let $\mathsf{n}(T)$ represent the number of nodes in a `BinaryTree` T. So,

$$\begin{aligned}
\mathsf{n}(0) = \mathsf{n}(1) &= 1 \\
\mathsf{n}((L,1,R)) = \mathsf{n}((L,0,R)) &= \mathsf{n}(L) + \mathsf{n}(R) + 1
\end{aligned}$$

Prove using structural induction that for all `BinaryTrees` T, $\mathsf{sum}(T) \le \mathsf{n}(T)$.

Solution:
Let $P(T)$ be $\mathsf{sum}(T) \le \mathsf{n}(T)$. We will prove that $P(T)$ holds for all `BinaryTrees` by structural induction.

Base cases: There are two basis steps: 0 and 1.
$\mathsf{sum}(0) = 0 \le 1 = \mathsf{n}(0)$. So, $P(0)$ holds.
$\mathsf{sum}(1) = 1 \le \mathsf{n}(1)$. So, $P(1)$ holds.

Inductive Hypothesis: Suppose $P(L)$ and $P(R)$ hold for arbitrary `BinaryTrees` L, R.

Inductive Step: *We want to show $P((L,0,R))$ and $P((L,1,R))$.*

$$\begin{aligned}
\mathsf{sum}((L,0,R)) &= \mathsf{sum}(L) + \mathsf{sum}(R) && \text{def. of sum} \\
&\le \mathsf{n}(L) + \mathsf{n}(R) && \text{IH} \\
&\le \mathsf{n}(L) + \mathsf{n}(R) + 1 \\
&= \mathsf{n}((L,0,R)) && \text{def. of n}
\end{aligned}$$

So, $P((L,0,R))$ holds.

$$\begin{aligned}
\mathsf{sum}((L,1,R)) &= \mathsf{sum}(L) + \mathsf{sum}(R) + 1 && \text{def. of sum} \\
&\le \mathsf{n}(L) + \mathsf{n}(R) + 1 && \text{IH} \\
&= \mathsf{n}((L,1,R)) && \text{def. of n}
\end{aligned}$$

So, $P((L,1,R))$ holds.

We conclude that $P(T)$ holds for all `BinaryTrees` T by structural induction.

5. Other things [14 points]

(a) (2 points) Consider the cosine function cos: $\mathbb{R} \to \mathbb{R}$.
Decide whether this function is one-to-one (injective) and whether it is onto (surjective).
◯ One-to-one/injective only
◯ Onto/surjective only
◯ Both
◯ Neither

Solution:

Neither

(b) (2 points) What if it had been defined as cos: $\mathbb{R} \to [-1, 1]$?
◯ One-to-one/injective only
◯ Onto/surjective only
◯ Both
◯ Neither

Solution:

Surjective

(c) (6 points) Prove the following statement using a proof by contrapositive:

For all integers n, if $5 \nmid n^2$, then $5 \nmid n$.

Solution:

We will prove this statement by contrapositive. Let n be an arbitrary integer, and suppose that $5|n$. By definition of divides, $n = 5k$ for some integer k. So, $n^2 = (5k)^2 = 25k^2 = 5(5k^2)$. By definition of divides, $5|n^2$. Since n was arbitrary, this shows that if $5|n$, then $5|n^2$ for all integers n. The contrapositive must also be true, so we have shown that for all integers n, if $5 \nmid n^2$, then $5 \nmid n$.

(d) [2 points] Suppose you are trying to prove the same statement in (c), but with a proof by contradiction. Write the ***first sentence*** of the proof.

Solution:

Suppose for the sake of contradiction that there exists an integer n such that $5 \nmid n^2$ and $5 \mid n$.

(e) [1 point] True or False: If a language can be represented with a regular expression, it can be recognized by an NFA.
◯ True
◯ False

Solution:

True

(f) [1 point] True or False: There are some regular languages that cannot be represented with a CFG.
◯ True
◯ False

Solution:

False

6. Models of Computation [15 points]

Let L be the language of strings over $\{0,\ 1\}$ where there is at least one occurrence of 1 AND at most two occurrences of 0.

Examples of strings that are in L: 1, 111, 11100, 101, 10110111
Examples of strings that are not in L: ϵ, 0, 00, 10001

(a) [4 points] Write a regular expression that represents L and a one sentence explanation of why your regular expression works.
Note: Don't worry about finding a short and simple answer – our regular expression is quite long.

Solution:

$$(1^*(0 \cup \varepsilon)11^*(0 \cup \epsilon)1^*) \cup (11^*(0 \cup \epsilon)1^*(0 \cup \epsilon)1^*) \cup (1^*(0 \cup \epsilon)1^*(0 \cup \epsilon)11^*)$$

The idea is that we force a 1 to appear either before, in between, or after the 0s. Each 0 position can also instead be an empty string to allow for scenarios where there are less than two 0s.

(b) [4 points] Write a CFG that matches L. Please indicate clearly what the start symbol of your CFG is.

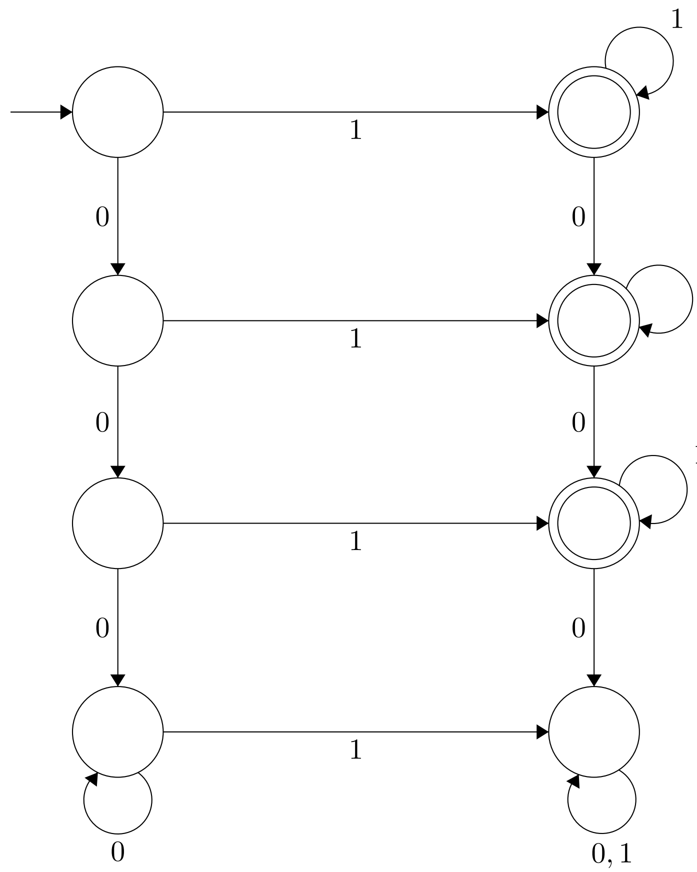
Solution:

$$\begin{aligned} S &\to 1YXYXY \mid YX1YXY \mid YXYX1Y \\ X &\to 0 \mid \epsilon \\ Y &\to 1Y \mid \epsilon \end{aligned}$$

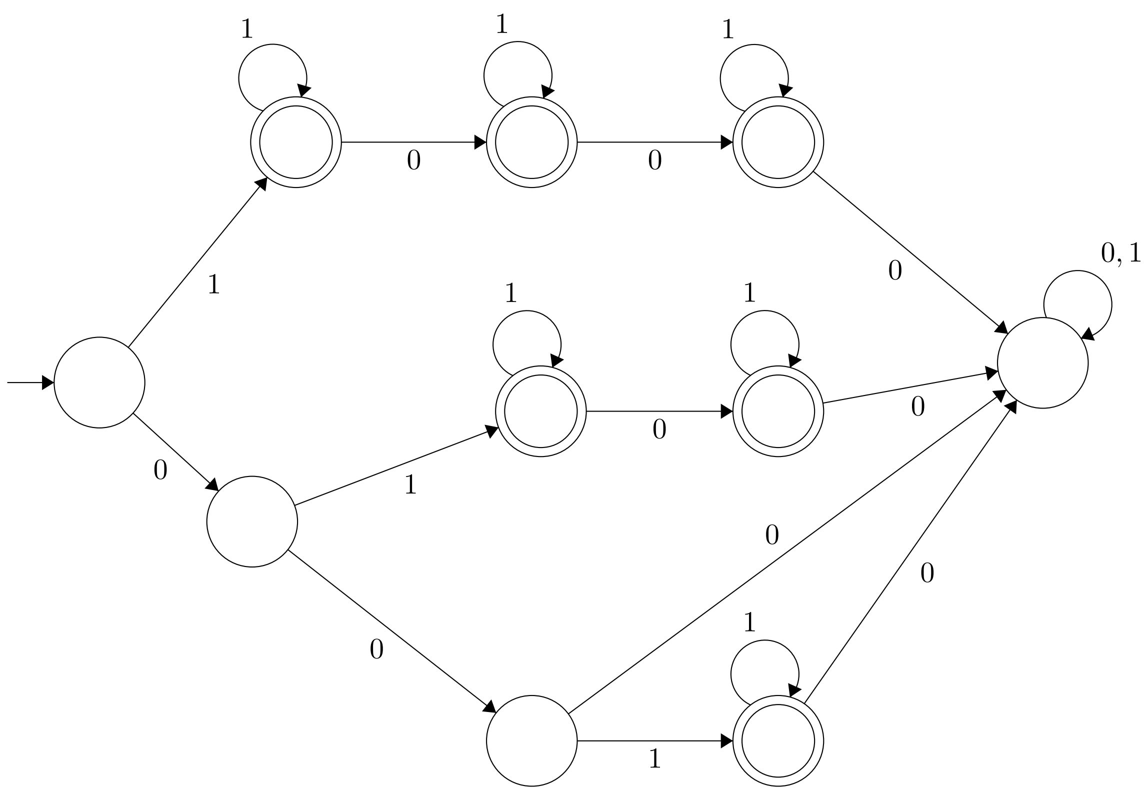
(c) [7 points] Write a DFA that accepts L.

Solution:

Solution 1 (cross product construction):



Solution 2 (matches the regular expression idea):



7. Irregularity Proof [15 points]
Let $\Sigma = \{a, b, c\}$. Let L be the following language:

$$L = \{w : w = a^k b^k c^{k-1} : k \geq 1\}$$

So, L contains strings of some integer $k \geq 1$ occurrences of a, followed by k b's, followed by $k - 1$ c's.

Examples of strings that are in L: ab, aabbc, aaabbbcc
Examples of strings that are not in L: abc, c, aaabcc

Prove that L is not regular.

Solution:
Suppose for the sake of contradiction that some DFA D accepts L.

Consider $S = \{a^n b^n : n \geq 1\}$. Since S contains infinitely many strings and D has a finite number of states, two strings of S must end up in the same state of D. Say those strings are $a^i b^i$ and $a^j b^j$ for some $i, j \geq 1$ such that $i \neq j$. Append c^{i-1} to both strings. The resulting strings are

$x = a^i b^i c^{i-1}$. Note that $x \in L$.
$y = a^j b^j c^{i-1}$. Note that $y \notin L$ since $i \neq j$, so $j - 1 \neq i - 1$.

Both x and y must end at the same state, but since $x \in L$ and $y \notin L$, the state must be both an accept and a reject state. This is a contradiction. So, there does not exist a DFA that recognizes L, which means L is not regular.

8. Grading Morale [1 point]

Draw a portrait of yourself on spring break!