CSE 390Z: Mathematics for Computation Workshop

Practice 311 Final

Name: ________________________________

UW ID: ______________________________

Instructions:

- This is a simulated practice final. You will not be graded on your performance on this exam.

- Nevertheless, please treat this as if it is a real exam. That means that you may not discuss with your neighbors, reference outside material, or use your devices during the next 50 minute period. (Note: On the real final, you would have 110 minutes)

- If you get stuck on a problem, consider moving on and coming back later. In the actual exam, there will likely be opportunity for partial credit.

- There are 7 problems on this exam, totaling 110 points.
1. **Predicate Translation** [15 points]

Let the domain of discourse be trees and places. Define the following predicates:

- Tree\(x\) := \(x\) is a tree
- InBloom\(x\) := \(x\) is in bloom
- CampusSpot\(x\) := \(x\) is a spot on campus
- Crowded\(x\) := \(x\) is crowded
- Located\((x, y)\) := \(x\) is located in \(y\)

For parts a-d, translate the sentence to predicate logic:

(a) [3 points] Not all spots on campus are crowded.

(b) [3 points] There is a tree located in every spot on campus.

(c) [3 points] There is more than one tree located in every spot on campus.

(d) [3 points] For any spot on campus, if a tree is in bloom there, it will be crowded.

(e) [3 points] Translate the **negation** of this statement to a natural English sentence:

\[ \forall x \forall y (\text{Tree}(x) \land \text{CampusSpot}(y) \land \text{Located}(x, y) \rightarrow \neg \text{InBloom}(x)) \]
2. Sets [15 points]

For any two sets $A$ and $B$, prove $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. 
3. **Induction I** [15 points]

Use induction to prove the following statement:

For all positive integers \( n \), \( n^3 + 2n \) is divisible by 3
4. **Induction II** [20 points]

**Recursive Definition of BinaryTrees:**
- Basis Steps: 0 is a BinaryTree and 1 is a BinaryTree
- Recursive Step: If $L, R$ are BinaryTrees, then $(L, 0, R)$ and $(L, 1, R)$ are also BinaryTrees

Intuitively, a BinaryTree is a binary tree where each node stores either a 0 or a 1.

**Recursive functions on BinaryTrees:**
The sum function returns the sum of all nodes in a BinaryTrees.

\[
\begin{align*}
\text{sum}(0) &= 0 \\
\text{sum}(1) &= 1 \\
\text{sum}((L, 0, R)) &= \text{sum}(L) + \text{sum}(R) \\
\text{sum}((L, 1, R)) &= \text{sum}(L) + \text{sum}(R) + 1
\end{align*}
\]

Let $n(T)$ represent the number of nodes in a BinaryTree $T$. So,

\[
\begin{align*}
n(0) &= n(1) = 1 \\
n((L, 1, R)) &= n((L, 0, R)) = n(L) + n(R) + 1
\end{align*}
\]

Prove using structural induction that for all BinaryTrees $T$, $\text{sum}(T) \leq n(T)$. 

5. Other things [14 points]

(a) (2 points) Consider the cosine function \( \cos : \mathbb{R} \rightarrow \mathbb{R} \).
Decide whether this function is one-to-one (injective) and whether it is onto (surjective).
- One-to-one/injective only
- Onto/surjective only
- Both
- Neither

(b) (2 points) What if it had been defined as \( \cos : \mathbb{R} \rightarrow [-1, 1] \)?
- One-to-one/injective only
- Onto/surjective only
- Both
- Neither

(c) (6 points) Prove the following statement using a proof by contrapositive:

For all integers \( n \), if \( 5 \nmid n^2 \), then \( 5 \nmid n \).

(d) [2 points] Suppose you are trying to prove the same statement in (c), but with a proof by contradiction. Write the first sentence of the proof.

(e) [1 point] True or False: If a language can be represented with a regular expression, it can be recognized by an NFA.
- True
- False

(f) [1 point] True or False: There are some regular languages that cannot be represented with a CFG.
- True
- False
6. Models of Computation [15 points]

Let \( L \) be the language of strings over \( \{0, 1\} \) where there is at least one occurrence of 1 AND at most two occurrences of 0.

Examples of strings that are in \( L \): 1, 111, 11100, 101, 10110111
Examples of strings that are not in \( L \): \( \epsilon \), 0, 00, 10001

(a) [4 points] Write a regular expression that represents \( L \) and a one sentence explanation of why your regular expression works.
   Note: Don’t worry about finding a short and simple answer – our regular expression is quite long.

(b) [4 points] Write a CFG that matches \( L \). Please indicate clearly what the start symbol of your CFG is.

(c) [7 points] Write a DFA that accepts \( L \).
7. Irregularity Proof [15 points]

Let $\Sigma = \{a, b, c\}$. Let $L$ be the following language:

$$L = \{w : w = a^k b^k c^{k-1} : k \geq 1\}$$

So, $L$ contains strings of some integer $k \geq 1$ occurrences of a, followed by $k$ b's, followed by $k - 1$ c's.

Examples of strings that are in $L$: ab, aabbc, aaabbbcc

Examples of strings that are not in $L$: abc, c, aaabcc

Prove that $L$ is not regular.
8. **Grading Morale** [1 point]

Draw a portrait of yourself on spring break!