CSE 390Z: Mathematics for Computation Workshop

Practice 311 Final

Name: ______________________________

UW ID: ______________________________

Instructions:

- This is a **simulated practice final**. You will **not** be graded on your performance on this exam.
- Nevertheless, please treat this as if it is a real exam. That means that you may not discuss with your neighbors, reference outside material, or use your devices during the next 50 minute period. (Note: On the real final, you would have 110 minutes)
- If you get stuck on a problem, consider moving on and coming back later. In the actual exam, there will likely be opportunity for partial credit.
- There are 7 problems on this exam, totaling 110 points.

1. Predicate Translation [15 points]

Let the domain of discourse be trees and places. Define the following predicates:

- Tree(x) := x is a tree
- InBloom(x) := x is in bloom
- CampusSpot(x) := x is a spot on campus
- Crowded(x) := x is crowded
- Located(x, y) := x is located in y

For parts a-d, translate the sentence to predicate logic:

(a) [3 points] Not all spots on campus are crowded.

(b) [3 points] There is a tree located in every spot on campus.

(c) [3 points] There is more than one tree located in every spot on campus.

(d) [3 points] For any spot on campus, if a tree is in bloom there, it will be crowded.

(e) [3 points] Translate the ***negation*** of this statement to a natural English sentence:

$$\forall x \forall y(\text{Tree}(x) \land \text{CampusSpot}(y) \land \text{Located}(x, y) \rightarrow \neg\text{InBloom}(x))$$

2. Sets [15 points]

For any two sets A and B, prove $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

3. Induction I [15 points]
Use induction to prove the following statement:

For all positive integers n, $n^3 + 2n$ is divisible by 3

4. Induction II [20 points]

Recursive Definition of BinaryTrees:

- Basis Steps: 0 is a `BinaryTree` and 1 is a `BinaryTree`
- Recursive Step: If L, R are `BinaryTrees`, then $(L, 0, R)$ and $(L, 1, R)$ are also `BinaryTrees`

Intuitively, a `BinaryTree` is a binary tree where each node stores either a 0 or a 1.

Recursive functions on BinaryTrees:
The sum function returns the sum of all nodes in a `BinaryTrees`.

$$\begin{aligned}
\mathsf{sum}(0) &= 0 \\
\mathsf{sum}(1) &= 1 \\
\mathsf{sum}((L, 0, R)) &= \mathsf{sum}(L) + \mathsf{sum}(R) \\
\mathsf{sum}((L, 1, R)) &= \mathsf{sum}(L) + \mathsf{sum}(R) + 1
\end{aligned}$$

Let $\mathsf{n}(T)$ represent the number of nodes in a `BinaryTree` T. So,

$$\begin{aligned}
\mathsf{n}(0) = \mathsf{n}(1) &= 1 \\
\mathsf{n}((L, 1, R)) = \mathsf{n}((L, 0, R)) &= \mathsf{n}(L) + \mathsf{n}(R) + 1
\end{aligned}$$

Prove using structural induction that for all `BinaryTrees` T, $\mathsf{sum}(T) \le \mathsf{n}(T)$.

5. Other things [14 points]

(a) (2 points) Consider the cosine function cos: $\mathbb{R} \to \mathbb{R}$.
Decide whether this function is one-to-one (injective) and whether it is onto (surjective).
◯ One-to-one/injective only
◯ Onto/surjective only
◯ Both
◯ Neither

(b) (2 points) What if it had been defined as cos: $\mathbb{R} \to [-1, 1]$?
◯ One-to-one/injective only
◯ Onto/surjective only
◯ Both
◯ Neither

(c) (6 points) Prove the following statement using a proof by contrapositive:

For all integers n, if $5 \nmid n^2$, then $5 \nmid n$.

(d) [2 points] Suppose you are trying to prove the same statement in (c), but with a proof by contradiction. Write the ***first sentence*** of the proof.

(e) [1 point] True or False: If a language can be represented with a regular expression, it can be recognized by an NFA.
◯ True
◯ False

(f) [1 point] True or False: There are some regular languages that cannot be represented with a CFG.
◯ True
◯ False

6. Models of Computation [15 points]

Let L be the language of strings over $\{0,\ 1\}$ where there is at least one occurrence of 1 AND at most two occurrences of 0.

Examples of strings that are in L: 1, 111, 11100, 101, 10110111
Examples of strings that are not in L: ϵ, 0, 00, 10001

(a) [4 points] Write a regular expression that represents L and a one sentence explanation of why your regular expression works.
Note: Don't worry about finding a short and simple answer – our regular expression is quite long.

(b) [4 points] Write a CFG that matches L. Please indicate clearly what the start symbol of your CFG is.

(c) [7 points] Write a DFA that accepts L.

7. Irregularity Proof [15 points]
Let $\Sigma = \{a, b, c\}$. Let L be the following language:

$$L = \{w : w = a^k b^k c^{k-1} : k \geq 1\}$$

So, L contains strings of some integer $k \geq 1$ occurrences of a, followed by k b's, followed by $k - 1$ c's.

Examples of strings that are in L: ab, aabbc, aaabbbcc
Examples of strings that are not in L: abc, c, aaabcc

Prove that L is not regular.

8. Grading Morale [1 point]

Draw a portrait of yourself on spring break!