0. Finish the Induction Proof

Consider the function \( f(n) \) defined for integers \( n \geq 1 \) as follows:
\[
\begin{align*}
f(1) &= 1 \\
f(2) &= 4 \\
f(3) &= 9 \\
f(n) &= f(n-1) - f(n-2) + f(n-3) + 2(2n-3) \quad \text{for } n \geq 4
\end{align*}
\]

Prove by strong induction that for all \( n \geq 1 \), \( f(n) = n^2 \).

**Complete the induction proof below.**

**Solution:**

1. Let \( P(n) \) be defined as "\( f(n) = n^2 \)." We will prove \( P(n) \) is true for all integers \( n \geq 1 \) by strong induction.

2. **Base Cases** \((n = 1, 2, 3)\):
   - \( n = 1 \): \( f(1) = 1 = 1^2 \).
   - \( n = 2 \): \( f(2) = 4 = 2^2 \).
   - \( n = 3 \): \( f(3) = 9 = 3^2 \)

   So the base cases hold.

3. **Inductive Hypothesis:** Suppose for some arbitrary integer \( k \geq 3 \), \( P(j) \) is true for \( 1 \leq j \leq k \).

4. **Inductive Step:**

   **Goal:** Show \( P(k+1) \), i.e. show that \( f(k+1) = (k + 1)^2 \).

   
   \[
   \begin{align*}
f(k + 1) &= f(k + 1 - 1) - f(k + 1 - 2) + f(k + 1 - 3) + 2(2(k + 1) - 3) & \text{Definition of } f \\
&= f(k) - f(k - 1) + f(k - 2) + 2(2k - 1) \\
&= k^2 - (k - 1)^2 + (k - 2)^2 + 2(2k - 1) & \text{By IH} \\
&= k^2 - (k^2 - 2k + 1) + (k^2 - 4k + 4) + 4k - 2 \\
&= (k^2 - k^2 + k^2) + (2k - 4k + 4k) + (-1 + 4 - 2) \\
&= k^2 + 2k + 1 \\
&= (k + 1)^2
\end{align*}
\]

   So \( P(k+1) \) holds.

5. **Conclusion:** So by strong induction, \( P(n) \) is true for all integers \( n \geq 1 \).
1. Prove the inequality
Prove by induction on $n$ that for all $n \in \mathbb{N}$ the inequality $(3 + \pi)^n \geq 3^n + n\pi \cdot 3^{n-1}$ is true.

Solution:
1. Let $P(n)$ be "$(3 + \pi)^n \geq 3^n + n\pi \cdot 3^{n-1}$". We will prove $P(n)$ is true for all $n \in \mathbb{N}$, by induction.
2. Base case ($n = 0$): $(3 + \pi)^0 = 1$ and $3^0 + 0 \cdot \pi \cdot 3^{-1} = 1$, since $1 \geq 1$, $P(0)$ is true.
3. Inductive Hypothesis: Suppose that $P(k)$ is true for some arbitrary integer $k \in \mathbb{N}$.
4. Inductive Step:

   Goal: Show $P(k+1)$, i.e. show $(3 + \pi)^{k+1} \geq 3^{k+1} + (k+1)\pi \cdot 3^{(k+1)-1} = 3^{k+1} + (k+1)\pi \cdot 3^{k}$

   \[
   (3 + \pi)^{k+1} = (3 + \pi)^k \cdot (3 + \pi)
   \geq (3^k + k\pi \cdot 3^{k-1}) \cdot (3 + \pi)
   = 3 \cdot 3^k + 3^k \pi + 3k^2 \pi + k3^{k-1} \pi^2
   = 3^{k+1} + 3^k \pi + k3^{k-1} \pi^2
   = 3^{k+1} + (k + 1)3^k \pi + k3^{k-1} \pi^2
   \geq 3^{k+1} + (k + 1)\pi \cdot 3^k
   \]

   (Factor out $(3 + \pi)$)  
   (By I.H., $(3 + \pi) \geq 0$)  
   (Distributive property)  
   (Simplify)  
   (Factor out $(k + 1)$)  
   $(k3^{k-1} \pi^2 \geq 0)$

5. So by induction, $P(n)$ is true for all $n \in \mathbb{N}$.
2. Inductively Odd

An 123 student learning recursion wrote a recursive Java method to determine if a number is odd or not, and needs your help proving that it is correct.

```java
public static boolean oddr(int n) {
    if (n == 0)
        return False;
    else
        return !oddr(n−1);
}
```

Help the student by writing an inductive proof to prove that for all integers \( n \geq 0 \), the method `oddr` returns True if \( n \) is an odd number, and False if \( n \) is not an odd number (i.e. \( n \) is even). You may recall the definitions \( \text{Odd}(n) := \exists x \in \mathbb{Z}(n = 2x + 1) \) and \( \text{Even}(n) := \exists x \in \mathbb{Z}(n = 2x) \); \!True = False and \!False = True.

**Solution:**

Let \( P(n) \) be "\( \text{oddr}(n) \) returns True if \( n \) is odd, or False if \( n \) is even". We will show that \( P(n) \) is true for all integers \( n \geq 0 \) by induction on \( n \).

**Base Case:** \( (n = 0) \)

0 is even, so \( P(0) \) is true if \( \text{oddr}(0) \) returns False, which is exactly the base case of `oddr`, so \( P(0) \) is true.

**Inductive Hypothesis:** Suppose \( P(k) \) is true for an arbitrary integer \( k \geq 0 \).

**Inductive Step:**

- **Case 1:** \( k + 1 \) is even.
  
  If \( k + 1 \) is even, then there is an integer \( x \) s.t. \( k + 1 = 2x \), so then \( k = 2x - 1 = 2(x - 1) + 1 \), so therefore \( k \) is odd. We know that since \( k + 1 > 0 \), \( \text{oddr}(k+1) \) should return \( \text{!oddr}(k) \). By the Inductive Hypothesis, we know that since \( k \) is odd, \( \text{oddr}(k) \) returns True, so \( \text{oddr}(k+1) \) returns \( \text{!oddr}(k) = \text{False} \), and \( k + 1 \) is even, therefore \( P(k+1) \) is true.

- **Case 2:** \( k + 1 \) is odd.
  
  If \( k + 1 \) is odd, then there is an integer \( x \) s.t. \( k + 1 = 2x + 1 \), so then \( k = 2x \) and therefore \( k \) is even. We know that since \( k + 1 > 0 \), \( \text{oddr}(k+1) \) should return \( \text{!oddr}(k) \). By the Inductive Hypothesis, we know that since \( k \) is even, \( \text{oddr}(k) \) returns False, so \( \text{oddr}(k+1) \) returns \( \text{!oddr}(k) = \text{True} \), and \( k + 1 \) is odd, therefore \( P(k+1) \) is true.

Then \( P(k + 1) \) is true for all cases. Thus, we have shown \( P(n) \) is true for all integers \( n \geq 0 \) by induction.
3. Strong Induction

Consider the function \( f(n) \) defined for integers \( n \geq 1 \) as follows:
\[
\begin{align*}
  f(1) &= 3 \\
  f(2) &= 5 \\
  f(n) &= 2f(n-1) - f(n-2)
\end{align*}
\]
Prove using strong induction that for all \( n \geq 1 \), \( f(n) = 2n + 1 \).

Solution:

Let \( P(n) \) be the claim that \( f(n) = 2n + 1 \). We will prove \( P(n) \) for all \( n \geq 1 \) by strong induction.

Base case:
\[
\begin{align*}
  f(1) &= 3 = 2 \times 1 + 1 \\
  f(2) &= 5 = 2 \times 2 + 1
\end{align*}
\]
So \( P(1) \) and \( P(2) \) are both true.

**Inductive Hypothesis:** Suppose for some arbitrary integer \( k \geq 2 \), \( P(2) \land \ldots \land P(k) \) hold.

**Inductive Step:**

**Goal:** Prove \( P(k+1) \), in other words, \( f(k+1) = 2(k+1) + 1 \)

\[
\begin{align*}
  f(k+1) &= 2f(k) - f(k-1) \\
  &= 2(2(k+1)) - (2k-1) \\
  &= 4k + 2 - (2k - 1) \\
  &= 2k + 3 \\
  &= 2(k+1) + 1
\end{align*}
\]

Therefore, \( f(k+1) = 2(k+1) + 1 \), so \( P(k+1) \) holds.

**Conclusion:** Therefore, \( P(n) \) holds for all numbers \( n \geq 1 \) by strong induction.
4. Strong Induction: Collecting Candy

A store sells candy in packs of 4 and packs of 7. Let \( P(n) \) be defined as "You are able to buy \( n \) packs of candy". For example, \( P(3) \) is not true, because you cannot buy exactly 3 packs of candy from the store. However, it turns out that \( P(n) \) is true for any \( n \geq 18 \). Use strong induction on \( n \) to prove this.

**Hint:** you’ll need multiple base cases for this - think about how many steps back you need to go for your inductive step.

**Solution:**

Let \( P(n) \) be defined as "You are able to buy \( n \) packs of candy". We will prove \( P(n) \) is true for all integers \( n \geq 18 \) by strong induction.

**Base Cases:** \( (n = 18, 19, 20, 21) \):

- \( n = 18 \): 18 packs of candy can be made up of 2 packs of 7 and 1 pack of 4 (\( 18 = 2 \times 7 + 1 \times 4 \)).
- \( n = 19 \): 19 packs of candy can be made up of 1 pack of 7 and 3 packs of 4 (\( 19 = 1 \times 7 + 3 \times 4 \)).
- \( n = 20 \): 20 packs of candy can be made up of 5 packs of 4 (\( 20 = 5 \times 4 \)).
- \( n = 21 \): 21 packs of candy can be made up of 3 packs of 7 (\( 21 = 3 \times 7 \)).

**Inductive Hypothesis:** Suppose for some arbitrary integer \( k \geq 21 \), \( P(18) \land \ldots \land P(k) \) hold.

**Inductive Step:**

**Goal:** Show \( P(k + 1) \), i.e. show that we can buy \( k + 1 \) packs of candy.

We want to buy \( k + 1 \) packs of candy. By the I.H., we can buy exactly \( k - 3 \) packs, so we can add another pack of 4 packs in order to buy \( k + 1 \) packs of candy, so \( P(k + 1) \) is true.

**Note:** How did we decide how many base cases to have? Well, we wanted to be able to assume \( P(k - 3) \), and add 4 to achieve \( P(k + 1) \). Therefore we needed to be able to assume that \( k - 3 \geq 18 \). Adding 3 to both sides, we needed to be able to assume that \( k \geq 21 \). So, we have to prove the base cases up to 21, that is: 18, 19, 20, 21.

Another way to think about this is that we had to use a fact from 4 steps back from \( k + 1 \) to \( k - 3 \) in the IS, so we needed 4 base cases.

**Conclusion:** So by strong induction, \( P(n) \) is true for all integers \( n \geq 18 \).
5. Structural Induction: Divisible by 4

Define a set $\mathcal{B}$ of numbers by:

- 4 and 12 are in $\mathcal{B}$
- If $x \in \mathcal{B}$ and $y \in \mathcal{B}$, then $x + y \in \mathcal{B}$ and $x - y \in \mathcal{B}$

Prove by induction that every number in $\mathcal{B}$ is divisible by 4.

**Complete the proof below:**

**Solution:**

Let $P(b)$ be the claim that $4 \mid b$. We will prove $P(b)$ is true for all numbers $b \in \mathcal{B}$ by structural induction.

**Base Case:**

- $4 \mid 4$ is trivially true, so $P(4)$ holds.

- $12 = 3 \cdot 4$, so $4 \mid 12$ and $P(12)$ holds.

**Inductive Hypothesis:** Suppose $P(x)$ and $P(y)$ for some arbitrary $x, y \in \mathcal{B}$.

**Inductive Step:**

| Goal: Prove $P(x + y)$ and $P(x - y)$ |

Per the IH, $4 \mid x$ and $4 \mid y$. By the definition of divides, $x = 4k$ and $y = 4j$ for some integers $k, j$. Then, $x + y = 4k + 4j = 4(k + j)$. Since integers are closed under addition, $k + j$ is an integer, so $4 \mid x + y$ and $P(x + y)$ holds.

Similarly, $x - y = 4k - 4j = 4(k - j) = 4(k + (-1 \cdot j))$. Since integers are closed under addition and multiplication, and $-1$ is an integer, we see that $k - j$ must be an integer. Therefore, by the definition of divides, $4 \mid x - y$ and $P(x - y)$ holds.

So, $P(t)$ holds in both cases.

**Conclusion:** Therefore, $P(b)$ holds for all numbers $b \in \mathcal{B}$. 
6. Structural Induction: CharTrees

Recursive Definition of CharTrees:

- Basis Step: Null is a CharTree
- Recursive Step: If $L, R$ are CharTrees and $c \in \Sigma$, then CharTree$(L, c, R)$ is also a CharTree

Intuitively, a CharTree is a tree where the non-null nodes store a char data element.

Recursive functions on CharTrees:

- The preorder function returns the preorder traversal of all elements in a CharTree.
  \[
  \text{preorder(Null)} = \varepsilon \\
  \text{preorder(CharTree}(L, c, R)) = c \cdot \text{preorder}(L) \cdot \text{preorder}(R)
  \]

- The postorder function returns the postorder traversal of all elements in a CharTree.
  \[
  \text{postorder(Null)} = \varepsilon \\
  \text{postorder(CharTree}(L, c, R)) = \text{postorder}(L) \cdot \text{postorder}(R) \cdot c
  \]

- The mirror function produces the mirror image of a CharTree.
  \[
  \text{mirror(Null)} = \text{Null} \\
  \text{mirror(CharTree}(L, c, R)) = \text{CharTree}(\text{mirror}(R), c, \text{mirror}(L))
  \]

- Finally, for all strings $x$, let the “reversal” of $x$ (in symbols $x^R$) produce the string in reverse order.

Additional Facts:
You may use the following facts:

- For any strings $x_1, \ldots, x_k$: $(x_1 \cdot \ldots \cdot x_k)^R = x_k^R \cdot \ldots \cdot x_1^R$
- For any character $c$, $c^R = c$

Statement to Prove:
Show that for every CharTree $T$, the reversal of the preorder traversal of $T$ is the same as the postorder traversal of the mirror of $T$. In notation, you should prove that for every CharTree, $T$: $(\text{preorder}(T))^R = \text{postorder}(\text{mirror}(T))$.

There is an example and space to work on the next page.
Let $T_i$ be the tree above.
$(T_i) = "abcd".$

This tree is $(T_i)$.
$((T_i)) = "dcba", \quad \text{"dcba" is the reversal of "abcd" so}$

$\text{[preorder}(T_i)\text{]}^R = \text{postorder}(\text{mirror}(T_i)) \text{ holds for } T_i$

Solution:
Let $P(T)$ be $\"[\text{preorder}(T)]^R = \text{postorder}(\text{mirror}(T))\"$. We show $P(T)$ holds for all CharTrees $T$ by structural induction.

Base case ($T = \text{Null}$): $\text{preorder}(T)^R = \varepsilon^R = \varepsilon = \text{postorder}(\text{Null}) = \text{postorder}(\text{mirror}(\text{Null}))$, so $P(\text{Null})$ holds.

Inductive hypothesis: Suppose $P(L) \land P(R)$ for arbitrary CharTrees $L, R$.

Inductive step:
We want to show $P(\text{CharTree}(L, c, R))$, i.e. $\text{[preorder}($\text{CharTree}(L, c, R)\text{)]}^R = \text{postorder}(\text{mirror}($\text{CharTree}(L, c, R)\text{)))}.$

Let $c$ be an arbitrary element in $\Sigma$, and let $T = \text{CharTree}(L, c, R)$

$$(T)^R = [c \cdot (L) \cdot (R)]^R$$

\hspace{1cm} \text{defn of preorder}

$$= (R)^R \cdot (L)^R \cdot c^R$$

\hspace{1cm} \text{Fact 1}

$$= (R)^R \cdot (L)^R \cdot c$$

\hspace{1cm} \text{Fact 2}

$$= ((R)) \cdot ((L)) \cdot c$$

\hspace{1cm} \text{by I.H.}

$$= (\text{CharTree}((R), c, (L)))$$

\hspace{1cm} \text{recursive defn of postorder}

$$= ((\text{CharTree}(L, c, R)))$$

\hspace{1cm} \text{recursive defn of mirror}

$$= (T))$$

\hspace{1cm} \text{defn of } T

So $P(\text{CharTree}(L, c, R))$ holds.

By the principle of induction, $P(T)$ holds for all CharTrees $T$. 