CSE 390Z: Mathematics for Computation Workshop

Week 3 Workshop Problems Solutions

Conceptual Review

(a) Inference Rules:

Introduce $\vee$: $\frac{A}{\therefore\ A\vee B,\ B\vee A}$ Eliminate $\vee$: $\frac{A\vee B\ ;\ \neg A}{\therefore\ B}$

Introduce $\wedge$: $\frac{A\ ;\ B}{\therefore\ A\wedge B}$ Eliminate $\wedge$: $\frac{A\wedge B}{\therefore\ A\ ,\ B}$

Direct Proof: $\frac{A\Rightarrow B}{\therefore\ A\rightarrow B}$ Modus Ponens: $\frac{A\ ;\ A\rightarrow B}{\therefore\ B}$

(b) Given $A \wedge B$, prove $A \vee B$

Solution:

1. $A \wedge B$ (Given)
2. A (Elim $\wedge$: 1.)
3. $A \vee B$ (Intro $\vee$: 2.)

Given $P \to R$, $R \to S$, prove $P \to S$.

1. $P \to R$ (Given)
2. $R \to S$ (Given)
   3.1 P (Assumption)
   3.2 R (Modus Ponens: 3.1, 1)
   3.3 S (Modus Ponens: 3.2, 2)
3. $P \to S$ (Direct Proof Rule; 3.1-3.3)

(c) What is a predicate, a domain of discourse, and a quantifier?

Solution:

Predicate: A function, usually based on one or more variables, that is true or false.

Domain of Discourse: The universe of values that variables come from.

Quantifier: A claim about when the predicate is true. There are two quantifiers. $\forall$ says that the claim is true for all values, and $\exists$ says there exists a value for which the claim is true.

(d) When translating to predicate logic, how do you restrict to a smaller domain in a "for all"? How do you restrict to a smaller domain in an "exists"?

Solution:

If we need to restrict something quantified by a "for all", we use **implication**. If we need to restrict something quantifies by an "exists", we use **and**.

For example, suppose the domain of discourse is all animals. We translate "all birds can fly" to $\forall x(\text{Bird}(x) \to \text{Fly}(x))$. We translate "there is a bird that can fly" to $\exists x(\text{Bird}(x) \wedge \text{Fly}(x))$.

(e) What are DeMorgan's Laws for Quantifiers?

Solution:

$\neg\forall x P(x) \equiv \exists x \neg P(x)$
$\neg\exists x P(x) \equiv \forall x \neg P(x)$

1. Predicate Logic Translations

Let the domain of discourse be all animals. Let $\texttt{Panda}(x)$ ::= "x is a panda" and $\texttt{KungFu}(x)$ ::= x knows kung fu. Translate the following statements to English.

(a) $\exists x(\neg\,\texttt{Panda}(x) \land \texttt{KungFu}(x))$

Solution:

There exists a non-panda that knows kung fu.

(b) $\forall x(\texttt{Panda}(x) \rightarrow \texttt{KungFu}(x))$

Solution:

All pandas know kung fu.

(c) $\neg\exists y(\texttt{Panda}(y) \land \neg\,\texttt{KungFu}(y))$

Solution:

There does not exist a panda that doesn't know kung fu. (Same meaning as the statement in part b!)

Your friend translated the sentence "there exists a panda who knows kung fu" to $\exists x(\text{Panda}(x) \rightarrow \text{KungFu}(x))$. This is wrong! Let's understand why.

(d) Use the Law of Implications to rewrite the translation without the $\rightarrow$.

Solution:

$\exists x(\neg\text{Panda}(x) \lor \text{KungFu}(x))$

(e) Translate the predicate from (d) back to English. How does this differ from the intended meaning?

Solution:

Translation: There exists an animal that is not a panda or that knows kung fu.

The difference: If there was even one non-panda animal in the universe (e.g. a turtle), this condition would be satisfied. Similarly, if there was even one animal in the universe that knew kung fu, this condition would be satisfied. So, this expression has a very different meaning than "there exists a panda who knows kung fu".

(f) This is a warning to be very careful when including an implication nested under an exists! It should almost always be avoided, unless there is a forall involved as well. (Nothing to write for this part).

2. Tricky Translations

Translate the following logical statements to natural English sentences. The domain of discourse is movies and actors. The following predicates are defined: $\text{Movie}(x)$ ::= x is a movie, $\text{Actor}(x)$ ::= x is an actor, $\text{Features}(x, y)$ ::= x features y.

(a) $\neg\exists x\exists y\exists z(\text{Movie}(x) \land \text{Actor}(y) \land \text{Actor}(z) \land y \neq z \land \text{Features}(x, y) \land \text{Features}(x, z))$

Solution:

No movie features two different actors.

(b) $\neg\forall x((\text{Movie}(x) \wedge \text{Feature}(x, \text{Daniel Radcliffe})) \rightarrow x = \text{Harry Potter})$

Solution:

Not all movies that feature Daniel Radcliffe are Harry Potter movies.
In other words, Daniel Radcliffe is in other movies besides Harry Potter.

(c) Below are logical expressions that look very similar, but only one is a correct translation of the sentence: "There is an actor that is featured in every movie". Find the correct translation and explain why the other options are wrong/nonsensical.

$\exists y \forall x((\text{Actor}(x) \wedge \text{Movie}(y)) \rightarrow \text{Features}(y, x))$

$\exists x \forall y(\text{Actor}(x) \wedge (\text{Movie}(y) \rightarrow \text{Features}(y, x))$

$\exists y \forall x(\text{Actor}(x) \wedge (\text{Movie}(y) \rightarrow \text{Features}(y, x))$

$\forall x \exists y(\text{Actor}(x) \wedge (\text{Movie}(y) \rightarrow \text{Features}(y, x))$

Solution:

The second one is correct.
The first option is incorrect because it does domain restriction incorrectly for the x. If there are no actors at all, the implication would still be vacuously true.
The third option is incorrect because it says that everything in the domain is an actor, which doesn't make sense.
The fourth option is incorrect for the same reason, it says that everything is an actor.

3. More Tricky Translations

Express the following sentences in predicate logic. The domain of discourse is movies and actors. You may use the following predicates: $\text{Movie}(x) ::= x$ is a movie, $\text{Actor}(x) ::= x$ is an actor, $\text{Features}(x, y) ::= x$ features y.

(a) Every movie features an actor.

Solution:

$\forall x(\text{Movie}(x) \rightarrow \exists y(\text{Actor}(y) \wedge \text{Features}(x, y)))$

(b) Not every actor has been featured in a movie.

Solution:

$\neg\forall x(\text{Actor}(x) \rightarrow \exists y(\text{Movie}(y) \wedge \text{Features}(y, x)))$

or, equivalently:

$\exists x(\text{Actor}(x) \wedge \forall y(\text{Movie}(y) \rightarrow \neg\text{Features}(y, x)))$

(c) All movies that feature Harry Potter must feature Voldermort.
Hint: You can use "Harry Potter" and "Voldemort" as constants that you can directly plug into a predicate.

Solution:

$\forall x((\text{Movie}(x) \wedge \text{Features}(x, \text{Harry Potter})) \rightarrow \text{Features}(x, \text{Voldemort}))$

(d) There is a movie that features **exactly one** actor.

Solution:

$\exists x \exists y(\text{Movie}(x) \wedge \text{Actor}(y) \wedge \text{Features}(x, y) \wedge \forall z((\text{Actor}(z) \wedge (z \neq y)) \rightarrow \neg\text{Features}(x, z)))$

4. Domains of Discourse

For the following, find a domain of discourse where the following statement is true and another where it is false. Note that for the arithmetic symbols to make sense, the domains of discourse should be sets of numbers.

(a) $\exists x(2x = 0)$

Solution:

True domain: Any set of numbers that includes 0; e.g. all natural numbers.
False domain: Any set of numbers that doesn't include 0; e.g. all integers greater than 0.

(b) $\forall x \exists y(x + y = 0)$

Solution:

True domain: Any set of numbers that includes additive inverses; e.g. all integers.
False domain: Any set of numbers that doesn't include additive inverses; e.g. all positive integers.

(c) $\exists x \forall y(x + y = y)$

Solution:

True domain: Any set of numbers that includes 0; e.g. all natural numbers (if $x = 0$, the statement holds for all y).
False domain: Any set of numbers that doesn't include 0; e.g. all integers greater than 0.

5. Negating Quantifiers

In the previous question, we translated the sentence "Not every actor has been featured in a movie" to predicate logic.

This was Kriti's translation: $\neg\forall x(\text{Actor}(x) \rightarrow \exists y(\text{Movie}(y) \wedge \text{Features}(y, x)))$

This was Tanush's translation: $\exists x(\text{Actor}(x) \wedge \forall y(\text{Movie}(y) \rightarrow \neg\text{Features}(y, x)))$

(a) Azita claims that Kriti and Tanush are both correct. Do you agree with Azita?

Solution:

Yes, both translations are correct.

(b) Use a chain of predicate logic equivalences to prove that the two translations are equivalent.
Hint: You may wish to use DeMorgan's Law for Predicates and the Law of Implication.

Solution:

$\neg\forall x(\text{Actor}(x) \to \exists y(\text{Movie}(y) \land \text{Features}(y,x)))$	
$\equiv \exists x(\neg(\text{Actor}(x) \to \exists y(\text{Movie}(y) \land \text{Features}(y,x))))$	DeMorgan's Law for Predicates
$\equiv \exists x(\neg(\neg\text{Actor}(x) \lor \exists y(\text{Movie}(y) \land \text{Features}(y,x))))$	Law of Implications
$\equiv \exists x(\neg\neg\text{Actor}(x) \land \neg\exists y(\text{Movie}(y) \land \text{Features}(y,x))))$	DeMorgan's Law
$\equiv \exists x(\text{Actor}(x) \land \neg\exists y(\text{Movie}(y) \land \text{Features}(y,x))))$	Double Negation
$\equiv \exists x(\text{Actor}(x) \land \forall y(\neg(\text{Movie}(y) \land \text{Features}(y,x)))))$	DeMorgan's Law for Predicates
$\equiv \exists x(\text{Actor}(x) \land \forall y(\neg\text{Movie}(y) \lor \neg\text{Features}(y,x)))))$	DeMorgan's Law
$\equiv \exists x(\text{Actor}(x) \land \forall y(\text{Movie}(y) \to \neg\text{Features}(y,x)))))$	Law of Implications

6. Translations with Integers

Translate the following English sentences to predicate logic. The domain is integers, and you may use $=$, $\neq$, and $>$ as predicates. Assume the predicates Prime, Composite, and Even have been defined appropriately.
Note: Composite numbers are ones that have at least 2 factors (the opposite of prime).

(a) 2 is prime.

Solution:

$\text{Prime}(2)$

(b) Every **positive** integer is prime or composite, but not both.

Solution:

$\forall x\ ((x > 0) \to (\text{Prime}(x) \oplus \text{Composite}(x)))$

OR

$\forall x\ ((x > 0) \to [(\text{Prime}(x) \land \neg\text{Composite}(x)) \lor (\neg\text{Prime}(x) \land \text{Composite}(x))])$

(c) There is **exactly one** even prime.

Solution:

$\exists x((\text{Even}(x) \land \text{Prime}(x) \land \forall y[(\text{Even}(y) \land \text{Prime}(y)) \to (y = x)])$

OR

$\exists x((\text{Even}(x) \land \text{Prime}(x) \land \forall y[(y \neq x) \to \neg(\text{Even}(y) \land \text{Prime}(y))])$

(d) 2 is the only even prime.

Solution:

$\forall x\ ((x = 2) \leftrightarrow \text{Prime}(x) \land \text{Even}(x))$

(e) Some, but not all, composite integers are even.

Solution:

$\exists x(\text{Composite}(x) \land \text{Even}(x)) \land \neg\forall x(\text{Composite}(x) \to \text{Even}(x))$

OR

$\exists x(\text{Composite}(x) \land \text{Even}(x)) \land \exists x(\text{Composite}(x) \land \neg\text{Even}(x))$

7. Formal Proofs: Modus Ponens

(a) Prove that given $p \to q$, $\neg s \to \neg q$, and p, we can conclude s.

Solution:

1. $p \to q$ (Given)
2. $\neg s \to \neg q$ (Given)
3. p (Given)
4. q (Modus Ponens; 1,3)
5. $q \to s$ (Contrapositive; 2)
6. s (Modus Ponens; 5,4)

(b) Prove that given $\neg(p \lor q) \to s$, $\neg p$, and $\neg s$, we can conclude q.

Solution:

1. $\neg(p \lor q) \to s$ (Given)
2. $\neg p$ (Given)
3. $\neg s$ (Given)
4. $\neg s \to \neg\neg(p \lor q)$ (Contrapositive; 1)
5. $\neg s \to (p \lor q)$ (Double Negation; 4)
6. $p \lor q$ (Modus Ponens; 3,5)
7. q (Elim $\lor$; 6,2)

8. Formal Proofs: Direct Proof Rule

(a) Prove that given $p \to q$, we can conclude $(p \land r) \to q$

Solution:

1. $p \to q$ (Given)
    - 2.1 $p \land r$ (Assumption)
    - 2.2 p (Elim $\land$; 2.1)
    - 2.3 q (Modus Ponens; 2.2, 1.)
2. $(p \land r) \to q$ (Direct proof rule; 2.1-2.3)

(b) Prove that given $p \lor q$, $q \to r$, and $r \to s$, we can conclude $\neg p \to s$.

Solution:

1. $p \lor q$ (Given)
2. $q \to r$ (Given)
3. $r \to s$ (Given)
   4.1 $\neg p$ (Assumption)
   4.2 q (Elim $\lor$; 1, 4.1)
   4.3 r (Modus Ponens; 4.2, 2)
   4.4 s (Modus Ponens; 4.3, 3)
4. $\neg p \to s$ (Direct proof rule; 4.1-4.4)

9. Challenge: Predicate Negation

Translate "You can fool all of the people some of the time, and you can fool some of the people all of the time, but you can't fool all of the people all of the time" into predicate logic. Then, negate your translation. Then, translate the negation back into English.

Hint: Let the domain of discourse be all people and all times, and let $P(x, y)$ be the statement "You can fool person x at time y". You can get away with not defining any other predicates if you use P.

Solution:

The original statement can thus be translated as

$$(\forall x \exists y P(x, y)) \land (\exists z \forall a P(z, a)) \land (\neg \forall b \forall c P(b, c))$$

The negation of this statement, in predicate logic, is

$$(\exists x \forall y \neg P(x, y)) \lor (\forall z \exists a \neg P(z, a)) \lor (\forall b \forall c P(b, c))$$

which in English translates to

"There are some people you can't ever fool, or all people have some time at which you can't fool them, or you can fool everyone at all times"